Integrable defects: an algebraic approach

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General frame

- Integrable defects (quantum level) impose severe constraints on relevant algebraic and physical quantities (such as on scattering amplitudes) (*Delfino, Mussardo, Simonetti, Konic, LeClair, ...*)

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- In integrable field theories a defect is introduced as discontinuity together with gluing conditions (*Bowcock, Corrigan, Zambon,...*), the integrability issue not systematically addressed.
- Aim is to develop a systematic algebraic means to investigate integrable filed theories with point like defects. Recent attempts by (*Habibullin, Kundu*), but integrability still open issue

Outline

- Introduce the discrete non-liner Schrodinger model. Recall the L-matrix and the associated classical quadratic algebra. Recall the local I.M. and the Lax pair construction based on purely algebraic grounds.
- Extract local integrals of motion, the relevant Lax pairs and the corresponding equations of motion in the presence of defect.
- Consider a consistent continuum limit of the model under consideration. First glimpse on the continuum model. First step in order to compare with earlier results (*Corrigan, Zambon*).
- Discussion on possible future applications of the proposed methodology.

The DNLS model

The DNLS Lax operator (*Kundu, Ragnisco*):

$$L(\lambda) = \begin{pmatrix} \lambda + \mathbb{N}_j & x_j \\ -X_j & 1 \end{pmatrix}$$

$\mathbb{N}_j = 1 - x_j X_j$ and $\{x_i,\ X_j\} = \delta_{ij}$. The L matrix satisfies the:

Classical quadratic algebra

$$\Big\{L_{an}(\lambda_1),\ L_{bm}(\lambda_2)\Big\} = \Big[r_{ab}(\lambda_1 - \lambda_2),\ L_{an}(\lambda_1)L_{bm}(\lambda_2)\Big]\delta_{nm}.$$

The classical r matrix satisfies the CYBE (*Semenov-Tian-Shansky*)

$$\boxed{[r_{12},\ r_{13}] + [r_{12},\ r_{23}] + [r_{13},\ r_{23}] = 0.}$$

The classical r-matrix in this case in the Yangian (*Yang*): $r(\lambda) = \frac{\mathcal{P}}{\lambda}$.

The DNLS model

The one-dimensional DNLS model; the generating function of all integrals of motion:

The transfer matrix

$$t(\lambda) = Tr_a \ T_a(\lambda) \quad \text{where} \quad T_a(\lambda) = L_{aN}(\lambda) L_{aN-1}(\lambda) \dots L_{a1}(\lambda).$$

T is the monodromy matrix also satisfying the quadratic algebra. Expansion of the log of the $t(\lambda)$ provides the *local* IM:

$$H_1 = \sum_{i=1}^{N} \mathbb{N}_i,$$

$$H_2 = -\sum_{i=1}^{N} x_{i+1} X_i - \frac{1}{2} \sum_{i=1}^{N} \mathbb{N}_i^2$$

$$H_3 = -\sum_{i=1}^{N} x_{i+2} X_i + \sum_{i=1}^{N} (\mathbb{N}_i + \mathbb{N}_{i+1}) x_{i+1} X_i + \frac{1}{3} \sum_{i=1}^{N} \mathbb{N}_i^3.$$

Lax pair formulation

Introduce the Lax pair $(L, \ \mathbb{A})$ for discrete integrable models, and the associated discrete:

Auxiliary linear problem

$$\psi_{j+1} = L_j \ \psi_j$$
$$\dot{\psi}_j = \mathbb{A}_j \ \psi_j.$$

From the latter equations one may immediately obtain the discrete zero curvature condition as a compatibility condition:

Zero curvature condition

$$\dot{L}_j = \mathbb{A}_{j+1} \ L_j - L_j \ \mathbb{A}_j.$$

Based on the underlying algebras construct the Lax pair.

Lax pair formulation

Necessary first to formulate, using the classical algebra:

$$\Big\{ \ln t(\lambda),\ L_{bj}(\mu) \Big\} = t^{-1}\ Tr_a\Big(T_a(N, j+1; \lambda)\ r_{ab}(\lambda-\mu)\ T_a(j,1;\lambda)\Big)\ L_{bj}(\mu) - L_{bj}(\mu)\ t^{-1}\ Tr_a\Big(T_a(N,j;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(j-1,1;\lambda)\Big).$$

Notation: $T(i,j;\lambda) = L_i(\lambda)...L_j(\lambda), \quad i > j.$
Recalling the classical equations of motion

$$\dot{L}_j(\mu) = \Big\{ \ln t(\lambda),\ L_j(\mu) \Big\},$$

and comparing the latter expressions we have:

The $\mathbb{A}$-operator

$$\mathbb{A}_j(\lambda,\mu) = t^{-1}(\lambda)\ tr_a\ \Big[T_a(N,j;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(j-1,1;\lambda)\Big]$$

Lax pair formulation

Substituting the Yangian r-matrix into the latter expression:

The $\mathbb{A}$-operator (Yangian)

$$\mathbb{A}_j(\lambda, \mu) = \frac{t^{-1}(\lambda)}{\lambda - \mu}\ T(j-1, 1; \lambda)\ T(N, j; \lambda).$$

Expand the latter expression in powers of $\frac{1}{\lambda}$ to obtain the Lax pairs associated to each one of the I.M.:

$$\mathbb{A}_j^{(1)}(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbb{A}_j^{(2)}(\mu) = \begin{pmatrix} \mu & x_j \\ -X_{j-1} & 0 \end{pmatrix},$$
$$\mathbb{A}_j^{(3)} = \begin{pmatrix} \mu^2 + x_j X_{j-1} & \mu x_j - x_j \mathbb{N}_j + x_{j+1} \\ -\mu X_{j-1} + X_{j-1}\mathbb{N}_{j-1} - X_{j-2} & -x_j X_{j-1} \end{pmatrix}.$$

Zero curvature condition leads to E.M.

The Defect

Introduce the defect located an site n:

$$\tilde{L}_{an} = \lambda + \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$$

the index n denotes the position of the defect on the one dimensional spin chain. The entries of the above $\tilde{L}$ matrix may be parameterized as:

$$\alpha_n = -\delta_n = \frac{1}{2}\cos(2\theta_n), \quad \beta_n = \frac{1}{2}\sin(2\theta_n)e^{2i\phi_n}, \quad \gamma_n = \frac{1}{2}\sin(2\theta_n)e^{-2i\phi_n}$$

It is shown via the quadratic algebra that $\alpha_n,\ \beta_n,\ \gamma_n,\ \delta_n$ satisfy:

$$\{\alpha_n,\ \beta_n\} = \beta_n$$
$$\{\alpha_n,\ \gamma_n\} = -\gamma_n$$
$$\{\beta_n,\ \gamma_n\} = 2\alpha_n$$

typical $\mathfrak{sl}_2$ exchange relations.

The Defect

Inserting the defect at the n site of the one dimensional lattice the corresponding monodromy matrix is expressed as:

Monodromy with defect

$$T_a(\lambda) = L_{aN}(\lambda) L_{aN-1}(\lambda) \dots \tilde{L}_{an}(\lambda) \dots L_{a1}(\lambda).$$

The $\tilde{L}$-operator is required to satisfy the same fundamental algebraic relation as the monodromy matrix, so

$$t(\lambda) = tr T(\lambda)$$

provides a family of Poisson commuting operators. Model *integrable* by construction.

Local Integrals of motion

Expansion of the log of the transfer matrix in powers of $\frac{1}{\lambda}$ provides the local I.M. Give here the first three:

$$\begin{aligned}
\mathcal{H}_1 &= \sum_{j\neq n} \mathbb{N}_j + \alpha_n \\
\mathcal{H}_2 &= -\sum_{j\neq n,n-1} x_{j+1}X_j - \frac{1}{2}\sum_{j\neq n} \mathbb{N}_j^2 - x_{n+1}X_{n-1} - \beta_n X_{n-1} + \gamma_n x_{n+1} - \frac{\alpha_n^2}{2} \\
\mathcal{H}_3 &= -\sum_{j\neq n,n\pm1} x_{j+1}X_{j-1} + \sum_{j\neq n,n-1} (\mathbb{N}_j + \mathbb{N}_{j+1})x_{j+1}X_j + \frac{1}{3}\sum_{j\neq n} \mathbb{N}_j^3 \\
&+ \tilde{x}_{n,n+1}\mathbb{N}_{n-1}X_{n-1} + \tilde{X}_{n,n-1}x_{n+1}\mathbb{N}_{n+1} + \alpha_n \tilde{x}_{n,n+1}X_{n-1} \\
&+ \alpha_n \tilde{X}_{n,n-1}x_{n+1} - \tilde{x}_{n,n+1}X_{n-2} - x_{n+2}\tilde{X}_{n,n-1} + \frac{{\alpha_n}^3}{3}
\end{aligned}$$

The associated Lax pair

The generic expression of Lax pairs, and expansion leads to: Lax pair $\mathbb{A}_j^{(1)}$ the same as in the bulk for all sites, $\mathbb{A}_j^{(2)}$ for $j \neq n,\ n+1$ is given by the bulk, but

$$\mathbb{A}_n^{(2)} = \begin{pmatrix} \mu & \beta_n + x_{n+1} \\ -X_{n-1} & 0 \end{pmatrix}, \qquad \mathbb{A}_{n+1}^{(2)} = \begin{pmatrix} \mu & x_{n+1} \\ \gamma_n - X_{n-1} & 0 \end{pmatrix}$$

$\mathbb{A}_j^{(3)}$ for $j \neq n,\ n \pm 1,\ n+2$ is given by the bulk and:

$$\mathbb{A}_{n-1}^{(3)} = \begin{pmatrix} \mu^2 + x_{n-1}X_{n-2} & \mu x_{n-1} + \tilde{x}_{n,n+1} - \mathbb{N}_{n-1}x_{n-1} \\ -\mu X_{n-2} - X_{n-3} + \mathbb{N}_{n-2}X_{n-2} & -X_{n-2}x_{n-1} \end{pmatrix}$$

$$\mathbb{A}_{n}^{(3)} = \begin{pmatrix} \mu^2 + \tilde{x}_{n,n+1}X_{n-1} & \mu\tilde{x}_{n,n+1} + x_{n+1} - \mathbb{N}_{n+1}x_{n+1} + \mathfrak{f} \\ -\mu X_{n-1} - X_{n-2} + \mathbb{N}_{n-1}X_{n-1} & -\tilde{x}_{n,n+1}X_{n-1} \end{pmatrix}$$

$$\mathbb{A}_{n+1}^{(3)} = \begin{pmatrix} \mu^2 + x_{n+1}\tilde{X}_{n,n-1} & \mu x_{n+1} + x_{n+2} - \mathbb{N}_{n+1}x_{n+1} \\ -\mu\tilde{X}_{n,n-1} - X_{n-1} + \mathbb{N}_{n-1}X_{n-1} + \mathfrak{g} & -\tilde{X}_{n,n-1}x_{n+1} \end{pmatrix}$$

$$\mathbb{A}_{n+2}^{(3)} = \begin{pmatrix} \mu^2 + x_{n+2}X_{n+1} & \mu x_{n+2} + x_{n+3} - \mathbb{N}_{n+2}x_{n+2} \\ -\mu X_{n+1} - \tilde{X}_{n,n-1} + \mathbb{N}_{n+1}X_{n+1} & -X_{n+1}x_{n+2} \end{pmatrix}$$

Equations of motion

For $j \neq n, \;\; n \pm 1, \;\; n \pm 2$ E.M. provided by the bulk equation, whereas for the points around the impurity are suitably modified. On the defect point in particular:

Zero curvature for defect point

$$\dot{\tilde{L}}_n(\lambda) = \mathbb{A}_{n+1}(\lambda) \; \tilde{L}_n(\lambda) - \tilde{L}_n(\lambda) \; \mathbb{A}_n(\lambda)$$

and the entailed equations of motion for the defect point are completely modified due to the presence of the defect degrees of freedom.

The continuum limit

Introduce the spacing parameter Δ in the L-matrix of the discrete NLS model as well as in the $\tilde{L}$ matrix of the defect:

$$L(\lambda) = \begin{pmatrix} 1 + \Delta\lambda - \Delta^2 xX & \Delta x \\ -\Delta X & 1 \end{pmatrix}$$

$$\tilde{L}(\lambda) = \Delta\lambda + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where we now define:

$$\alpha = -\delta = \frac{1}{2}\cos(2\Delta\theta), \qquad \beta = \frac{1}{2}\sin(2\Delta\theta)e^{2i\phi}, \qquad \gamma = \frac{1}{2}\sin(2\Delta\theta)e^{-2i\phi},$$

also

$$\theta e^{2i\phi} = y, \qquad \theta e^{-2i\phi} = Y,$$

The continuum limit

Let us first introduce the following notation. In particular, we set:

Identifications

$$x_j \;\to\; x^-(x), \qquad X_j \;\to\; X^-(x), \qquad 1 \le j \le n-1, \qquad x \in (-\infty,\; x_0)$$
$$x_j \to x^+(x), \qquad X_j \to X^+(x), \qquad n+1 \le j \le N, \qquad x \in (x_0,\; \infty).$$

where x_0 is the defect position in the continuum theory. To perform the continuum limit we bear in mind:

The limit

$$\Delta \sum_{j=1}^{n-1} f_j \;\to\; \int_{-\infty}^{x_0^-} dx\; f^-(x)$$
$$\Delta \sum_{j=n+1}^{N} f_j \;\to\; \int_{x_0^+}^{\infty} dx\; f^+(x).$$

Also: $f_{j+1} \to f(x+\Delta)$

The continuum limit

The continuum limit of the first integral of motion is then given as:

The first carge

$$\mathcal{H}^{(1)} = -\int_{-\infty}^{x_0^-} dx\ x^-(x)X^-(x) - \int_{x_0^+}^{\infty} dx\ x^+(x)X^+(x).$$

The first integral proportional to Δ, whereas the second one of order Δ^2. The respective continuum quantity reads as:

The second charge

$$\begin{aligned}\mathcal{H}^{(2)} &= -\int_{-\infty}^{x_0^-} dx\ x^{-'}(x)X^-(x) - \int_{x_0^+}^{\infty} dx\ x^{+'}(x)X^+(x) + \frac{1}{2}y(x_0)Y(x_0) \\ &+ x^-(x_0)X^-(x_0) - x^+(x_0)X^-(x_0) + x^+(x_0)Y(x_0) - y(x_0)X^-(x_0)\end{aligned}$$

the prime denotes derivative with respect to x.

The continuum limit

Consider the identifications:

$$L_n \to 1 + \Delta \mathbb{U}(x), \quad \mathbb{A}_n \to \mathbb{V}(x), \quad \mathbb{A}_{n+1} \to \mathbb{V}(x+\Delta)$$

The discrete zero curvature condition:

$$\dot{L}_j = \mathbb{A}_{j+1}\ L_j - L_j\ \mathbb{A}_j.$$

Then takes the familiar continuum form:

Continuum zero curvature

$$\dot{\mathbb{U}} - \mathbb{V}' + \Big[\mathbb{U},\ \mathbb{V}\Big] = 0.$$

We have kept terms proportional to Δ in the discrete zero curvature condition.

The continuum limit

The Lax pair associated to the first integral coincides with the bulk one. The Lax pair associated to the second integral of motion is given by:

$$\mathbb{V}^{(2)}(\mu,\ x) = \begin{pmatrix} \mu & x^-(x) \\ -X^-(x) & 0 \end{pmatrix} \qquad x \in (-\infty,\ x_0^-],$$
$$\mathbb{V}^{(2)}(\mu,\ x) = \begin{pmatrix} \mu & x^+(x) \\ -X^+(x) & 0 \end{pmatrix} \qquad x \in (x_0^+,\ \infty)$$
$$\mathbb{V}^{(2)}(\mu,\ x_0) = \begin{pmatrix} \mu & x^+(x_0) + y(x_0) \\ -X^-(x_0) & 0 \end{pmatrix} ,$$
$$\mathbb{V}^{(2)}(\mu,\ x_0^+) = \begin{pmatrix} \mu & x^+(x_0) \\ Y(x_0) - X^-(x_0) & 0 \end{pmatrix} .$$

Due to continuity requirements at the points $x_0^+,\ x_0^-$, we end up with the following sewing conditions associated to the defect point:

Sewing conditions

$$y(x_0) = x^-(x_0) - x^+(x_0),$$
$$Y(x_0) = X^-(x_0) - X^+(x_0).$$

The continuum limit

- The continuity argument successfully applied to the points around the defect; a discontinuity (jump) is observed on the defect point. At x_0 (i.e. $L \rightarrow \tilde{L}$), leading to discontinuity in the zero curvature condition at x_0.
- It is straightforward to show that if the sewing conditions are valid then:

Commutativity

$$\{\mathcal{H}_1,\ \mathcal{H}_2\} = 0$$

- A first indication of the preservation of the integrability in the continuum case. In the discrete an ultra local algebra; t in the continuum limit possibly a generalized non-ultra local algebra to efficiently describe the point like defect at x_0.

Systematic construction

Recall

$$L_{ai} = 1 + \delta \mathbb{U}_{ai} + \mathcal{O}(\delta^2) \ ,$$

Then the monodromy matrix is expanded as:

$$T_a = 1 + \delta \sum_i \mathbb{U}_{ai} + \delta^2 \sum_{i<j} \mathbb{U}_{a_i} \mathbb{U}_{aj} + \ldots \ .$$

Which leads to the familiar continuum expression

The continuum monodromy

$$\mathcal{T} = P \exp\left(\int_0^A dx \ \mathbb{U}(x) \right)$$

Systematic construction

Then the discrete monodromy matrix in the presence of defect:

$$T_a(\lambda) = L_{aN}(\lambda) \dots \tilde{L}_{an}(\lambda) \dots L_{a1}(\lambda)$$

according to previous analysis T will be formally expressed at the continuum limit:

The defect monodromy

$$\mathcal{T}(\lambda) = P\exp\Big(\int_0^{x_0^-} dx\ \mathbb{U}^-(x)\Big)\ \tilde{\mathcal{L}}(\lambda)\ P\exp\Big(\int_{x_0^+}^{A} dx\ \mathbb{U}^+(x)\Big)$$

Algebraic constraints on $\tilde{\mathcal{L}}$? Non ultra-locality ensued?

Discussion

- Examine higher integrals of motion and check their involution in the continuum limit. Check also the consistency of higher gluing conditions (higher derivatives are involved).

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- Systematically establish the underlying Poisson structure governing this type of models. Integrability then naturally follows.
- Extend the study to other discrete integrable models associated e.g. to (an)isotropic Heisenberg chains, and higher rank generalizations.
- At the quantum level: derive the associated transmission amplitudes via the Bethe ansatz equations.