Analytical Bethe ansatz for integrable open spin chains

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To Daniel

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The programm

• Treat classes of integrable spin chains with open boundaries. Two main parts:
(i) $osp(n|m)$, ($so(n)$, $sp(m)$).
(ii) $gl(n)$ two distinct types of b.c. (SP. and SNP), deal also with any irrep of $\mathcal{Y}(gl(n))$.

• Main aims classification of solutions of the RE $\longrightarrow$ derive the corresponding open spin chain.

• Then find the spectrum and BAE via the analytical BA method (the $gl(n)$ case results obtained for any rep.). Also boundary scattering via the study in the thermodynamic limit.

A. $osp(n|m)$, $so(n)$, $sp(m)$

The R matrix of the corresponding (super) Yangian,

$$R(\lambda) = \lambda(\lambda + i\rho)\mathbb{I} + (\lambda + i\rho)\mathcal{P} - \lambda Q$$

$$\mathcal{P}^2 = \mathbb{I}, \quad Q^2 = \Theta_0(n-m)Q, \quad \mathcal{P}Q = Q\mathcal{P} = \Theta_0 Q$$
$$\rho = \Theta_0 \frac{-n+m-2}{2}, \quad \Theta_0 = \pm 1,$$

Solution of the Yang-Baxter equation (Baxter '72)

$$R_{12}(\lambda_1 - \lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

Also satisfies

- Unitarity $R(\lambda)\ R(-\lambda) \propto \mathbb{I}$

- Crossing $V_1\ R_{12}(-\lambda - i\rho)^{t_2}\ V_1 = R_{12}(\lambda)$.

The open spin chain

Introduce the K matrix, solution of the reflection equation (Cherednik '84)

$$\begin{aligned} & R_{12}(\lambda_1-\lambda_2)\ K_1(\lambda_1)\ R_{21}(\lambda_1+\lambda_2)\ K_2(\lambda_2) \\ = & \ K_2(\lambda_2)\ R_{12}(\lambda_1-\lambda_2)\ K_1(\lambda_1)\ R_{21}(\lambda_1-\lambda_2) \end{aligned}$$

Then (for any K) (Sklyanin '88)

$$\mathcal{T}(\lambda)\ = T(\lambda)\ K(\lambda)\ T^{-1}(-\lambda), \qquad T(\lambda) = R_{0N}(\lambda)\cdots R_{01}(\lambda)$$

$T \to$ tensor rep. of (super) Yangian, $\mathcal{T} \to$ tensor rep. of RA.

Define the transfer matrix

$$t(\lambda) = Tr_0\{K^+(\lambda)\ \mathcal{T}(\lambda)\}$$

and

$$[t(\lambda),\ t(\lambda')] = 0$$

Integrability ensured.

Diagonal solutions of the RE

D1. $K(\lambda) = diag(\underbrace{\alpha, \ldots, \alpha}_{k}, \underbrace{\beta, \ldots, \beta}_{k})$ for $so(2k)$, $sp(2k)$

$\alpha = -\lambda + i\xi, \quad \beta = \lambda + i\xi$

D2. $K(\lambda) = diag(\alpha, \beta, \ldots, \beta, \gamma)$ (no for $sp(n)$)

$\alpha = \frac{-\lambda + i\xi_1}{\lambda + i\xi_1}, \quad \beta = 1, \quad \gamma = \frac{-\lambda + i\xi_n}{\lambda + i\xi_n}, \quad \xi_1 + \xi_n = \rho - 1$

D3. $K(\lambda) = diag(\underbrace{\alpha, \ldots, \alpha}_{p}, \underbrace{\beta, \ldots, \beta}_{n-2p}, \underbrace{\alpha, \ldots, \alpha}_{p})$

$\alpha = -\lambda + i\xi, \quad \beta = \lambda + i\xi$, and $\xi = \frac{n}{4} - \rho$ fixed

D4. $K(\lambda) = diag(\alpha, \beta, \gamma, \delta)$, for $so(4)$

$\alpha = (-\lambda + i\xi_+)(-\lambda + i\xi_-), \quad \beta = (\lambda + i\xi_+)(-\lambda + i\xi_-)$
$\gamma = (-\lambda + i\xi_+)(\lambda + i\xi_-), \quad \delta = (\lambda + i\xi_+)(\lambda + i\xi_-)$

Analytical Bethe ansatz

Objective: find spectrum and BAE via analytical BA (Reshetikhin '83, Mezincescu, Nepomechie '92), exploit requirements

1. Reference state

2. Crossing symmetry $t(\lambda) = t(-\lambda - i\rho)$

3. Constraints from fusion

4. Analyticity requirements

5. Symmetry, from the asymptotics $\rightarrow t(\lambda)$ associated to the certain conserved quantities (e.g diagonal gen of the algebra).

There exists an obvious reference state:

$$|\Omega\rangle = \bigotimes_{i=1}^{N} \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}_i$$

The ref. state eigenstate of t with eigenvalue

$$\Lambda^0(\lambda) = a^{2N} g_0(\lambda) + \sum_i b^{2N} g_1(\lambda) + c^{2N} g_{n+m-1}(\lambda)$$

g_i depend on the choice of boundaries.

Conjecture:

$$\Lambda(\lambda) = a^{2N} g_0 D_0 + \sum_i b^{2N} g_i D_i + c^{2N} g_{n+m-1} D_{n+m-2}$$

D_i to be determined by implementing the aforementioned constraints.

Bethe ansatz equations

Determine D_i and spectrum, analyticity requirements $\rightarrow$ BAE:
e.g. for $so(2k+1)$ ($K = \mathbb{I}$), k equations (no. of roots)

$$e_1^{2N+1}(\lambda_i^{(1)}) = -\prod_{j=1}^{M^{(1)}} \hat{e}_2(\lambda_i^{(1)}; \lambda_j^{(1)}) \prod_{j=1}^{M^{(2)}} \hat{e}_{-1}(\lambda_i^{(1)}; \lambda_j^{(2)})$$

$$e_1(\lambda_i^{(l)}) = -\prod_{j=1}^{M^{(l)}} \hat{e}_2(\lambda_i^{(l)}; \lambda_j^{(l)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(l+\tau)}} \hat{e}_{-1}(\lambda_i^{(l)}; \lambda_j^{(l+\tau)})$$

$l \in \{2. \dots, k-1\}$

$$e_{1/2}(\lambda_i^{(k)}) = -\prod_{j=1}^{M^{(k)}} \hat{e}_1(\lambda_i^{(k)}; \lambda_j^{(k)}) \prod_{j=1}^{M^{(k-1)}} \hat{e}_{-1}(\lambda_i^{(k)}; \lambda_j^{(k-1)})$$

where

$$e_n(\lambda) = \frac{\lambda + \frac{in}{2}}{\lambda - \frac{in}{2}}, \quad \hat{e}_n(\lambda_i; \lambda_j) = e_n(\lambda_i - \lambda_j) e_n(\lambda_i + \lambda_j)$$

for $so(2k)$ similar the last three equations are modified according to Dynkin diagram, similarly for $sp(2k)$ last 2 equ. modified:

$$e_1(\lambda_i^{(k-1)}) = - \prod_{j=1}^{M^{(k-1)}} \hat{e}_2(\lambda_i^{(k-1)}; \lambda_j^{(k-1)}) \prod_{j=1}^{M^{(k-2)}} \hat{e}_{-1}(\lambda_i^{(k-1)}; \lambda_j^{(k)}) \times \prod_{j=1}^{M^{(k)}} \hat{e}_{-2}(\lambda_i^{(k-1)}; \lambda_j^{(k)})$$

$$e_2(\lambda_i^{(k)}) = - \prod_{j=1}^{M^{(k)}} \hat{e}_4(\lambda_i^{(k)}; \lambda_j^{(k)}) \prod_{j=1}^{M^{(k-1)}} \hat{e}_{-2}(\lambda_i^{(k)}; \lambda_j^{(k-1)})$$

Comments

- From asymptotics: $M^{(i)}$ associated to the diagonal gens. of $so(n)$, ($sp(n)$).

- BAE 'doubled' compared to the bulk ones found in (Ogievetsky, Reshetikhin, Wiegmann '87).

- For $osp(n|m)$ the equ corresponding to the fermionic root changes as well.

- Different diagonal boundaries $\rightarrow$ multipl. factors in the LHS of BAE.

- Boundary scattering computed for various diagonal b.c., and bulk scattering computed compared with previous results (Ogievetsky, Reshetikhin, Wiegmann '87)

B. The $gl(n)$ case

The R matrix

$$R(\lambda) = \lambda + i\mathcal{P}$$

R satisfies unitarity but not crossing, therefore define:

$$\bar{R}_{12}(\lambda) = V_1 \; R_{12}(-\lambda - i\rho)^{t_2} \; V_1$$

$\rightarrow$ 2 types of b.c. (SP and SNP)

(I) SP deal with solutions of the usual reflection equation (reflection algebra)(e.g. de Vega, Gonzalez-Ruiz '94)

(II) SNP (twisted Yangian) (classically intro. Bowckock, Corrigan, Dorey, Rietdijk '95, quantum A.D. '00)

$$\begin{aligned} & R_{12}(\lambda_1 - \lambda_2) \; K_1(\lambda_1) \; \bar{R}_{21}(\lambda_1 + \lambda_2) \; K_2(\lambda_2) \\ = \; & K_2(\lambda_2) \; \bar{R}_{12}(\lambda_1 - \lambda_2) \; K_1(\lambda_1) \; R_{21}(\lambda_1 - \lambda_2) \end{aligned}$$

The Yangian $\mathcal{Y}(gl(n))$

Let $\mathcal{L}(\lambda) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{Y}$, satisfying

$$R_{12}(\lambda_1 - \lambda_2)\ \mathcal{L}_{13}(\lambda_1)\ \mathcal{L}_{23}(\lambda_2) = \mathcal{L}_{23}(\lambda_2)\ \mathcal{L}_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

Via $\mathcal{Y} \ \rightarrow \ gl(n)$ simply write:

$$\mathcal{L}(\lambda) = \sum_{i,j=1}^{n} E_{ij} \otimes \mathcal{L}_{ij}(\lambda), \qquad \mathcal{L}_{ij} \in gl(n)$$

$\Delta : \mathcal{Y} \rightarrow \mathcal{Y} \otimes \mathcal{Y}$

$$\Delta(\mathcal{L}_{ij}(\lambda)) = \sum_{k=1}^{n} \mathcal{L}_{ik}(\lambda) \otimes \mathcal{L}_{kj}(\lambda).$$

Tensor representation of $\mathcal{Y}$

$$T_0(\lambda) = \mathcal{L}_{0N}(\lambda) \ldots \mathcal{L}_{01}(\lambda) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{Y}^{\otimes N}$$

auxiliary space repr. via evaluation rep.

Useful to define:

$$\hat{\mathcal{L}}_{0i}(\lambda) = V_0^{t_0}\ \mathcal{L}_{0i}^{t_0}(-\lambda - i\rho)\ V_0^{t_0}$$

Derive tensor representations of RA and twisted Yangian:

(I) SP (reflection algebra)

$$\mathcal{T}(\lambda) = T(\lambda)\ K(\lambda)\ \hat{T}(\lambda), \qquad \hat{T}(\lambda) = T^{-1}(-\lambda)$$

(II) SNP (twisted Yangian)

$$\mathcal{T}(\lambda) = T(\lambda)\ K(\lambda)\ \hat{T}(\lambda), \quad \hat{T}(\lambda) = \hat{\mathcal{L}}_{01} \dots \hat{\mathcal{L}}_{0N}.$$

The transfer matrix:

$$t(\lambda) = Tr_0\Big\{K^+(\lambda)\ \mathcal{T}(\lambda)\Big\} \in \mathcal{Y}^{\otimes N}$$

Depending on V, t for SNP is associated to $so(n)$ or $sp(n)$

Focus on diagonal solutions

(SP) $K = \mathbb{I}$ $gl(n)$ symmetry or

$$K(\lambda) = diag(\underbrace{\alpha, \ldots, \alpha}_{k}, \underbrace{\beta, \ldots, \beta}_{n-k})$$

$\alpha = -\lambda + i\xi, \quad \beta = \lambda + i\xi$ (de Vega, Gonzalez-Ruiz '94)

$gl(n-k) \otimes gl(k)$ symmetry.

(SNP) $K = \mathbb{I}$ $so(n)$ or $sp(k)$ symmetry

Apply analytical BA.
Reference state (highest weight):

$$T_{ij},\ \hat{T}_{ij}\ u^+ = 0, \qquad i > j$$

$$T_{kk}(\lambda)\ u^+ = P_k(\lambda)\ u^+, \qquad \hat{T}_{kk}(\lambda) = \hat{P}_k(\lambda)\ u^+,$$

P_k Drinfeld Polynomials introduced for the classification of reps of $\mathcal{Y}$.

Using the highest weight and the underlying algebra (RA or TY) $\rightarrow$ u^+ eigenstate of t, for both b.c., and also compute eigenvalue.

Pseudo vacuum eigenvalue:

$$\Lambda^0(\lambda) = \sum_{k=1}^{n} g_k(\lambda)\ \beta_k(\lambda)$$

g_k depend on b.c., β_k depend on rep.

Conjecture:

$$\Lambda(\lambda) = \sum_{k=1}^{n} g_k(\lambda)\ \beta_k(\lambda)\ D_k(\lambda)$$

Via the analytical BA constraints determine D_k and spectrum.

Note crossing holds for SNP only, not for SP. For SP one needs an 'auxiliary' $\bar{t}$: $\bar{t}(\lambda) = t(-\lambda - i\rho)$.

Bethe ansatz equations

(SP) $n-1$ equations

$$\frac{g_l(\lambda_i^{(l)}-\frac{l}{2})\beta_l(\lambda_i^{(l)}-\frac{l}{2})}{g_{l+1}(\lambda_i^{(l)}-\frac{l}{2})\beta_{l+1}(\lambda_i^{(l)}-\frac{l}{2})} = -\prod_{j=1}^{M^{(l)}} \hat{e}_2(\lambda_i^{(l)};\lambda_j^{(l)}) \prod_{\tau=\pm1} \prod_{j=1}^{M^{(l+\tau)}} \hat{e}_{-1}(\lambda_i^{(l)};\lambda_j^{(l+\tau)})$$
$$l \in \{1,\dots n-1\}$$

(SNP) Similarly as in SP. Only the last equation is modified:
(i) $n = 2k+1$

$$\frac{g_k(\lambda_i^{(k)}-\frac{k}{2})\beta_k(\lambda_i^{(k)}-\frac{k}{2})}{g_{k+1}(\lambda_i^{(k)}-\frac{k}{2})\beta_{k+1}(\lambda_i^{(k)}-\frac{k}{2})} = -\prod_{j=1}^{M^{(k)}} \hat{e}_2(\lambda_i^{(k)};\lambda_j^{(k)})e_{-1}(\lambda_i^{(k)};\lambda_j^{(k)}) \times \prod_{j=1}^{M^{(k-1)}} \hat{e}_{-1}(\lambda_i^{(k)};\lambda_j^{(k-1)})$$

BAE as in $osp(1|2k)$.

(ii) $n = 2k$

$$\frac{g_k(\lambda_i^{(k)}-\frac{k}{2})\beta_k(\lambda_i^{(k)}-\frac{k}{2})}{g_{k+1}(\lambda_i^{(k)}-\frac{k}{2})\beta_{k+1}(\lambda_i^{(k)}-\frac{k}{2})} = -\prod_{j=1}^{M^{(k)}} \hat{e}_2(\lambda_i^{(k)};\lambda_j^{(k)}) \prod_{j=1}^{M^{(k-1)}} \hat{e}^2_{-1}(\lambda_i^{(k)};\lambda_j^{(k-1)})$$

The boundaries modify the bulk behavior in SNP. In SP BAE ‘doubled compared to the bulk case.

Discussion

• The super $sl(n|m)$ case has been also studied in the same spirit.

• Non diagonal boundaries also treated for $gl(n)$ via simple gauge transformations (SP). BAE preserve their structure

• The $U_q(gl_n)$ case work in progress for both types of b.c.

• Non-diagonal boundaries (q-deformed case). Symmetry plays crucial role (boundary non-local charges ((SNP) Delius, Mackay '02, (SP) A.D. '04)). Ref. state via local gauge transformation and symmetry arguments $\longrightarrow$ spectrum.