Integrable quantum spin chains and their classical continuous counterparts

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Motivation

- Integrable spin chains interesting exactly solvable models appear e.g. in statistical mechanics with prototype models the Heisenberg chain and generalizations (*Baxter '70s, Faddeev, Takhtajan '80s*...).
- Integrable models arise within the AdS/CFT correspondence (*Minahan, Zarembo '03*) both from the string theory and the gauge theory point of view. Also integrable spin chain appear in the high energy QCD scattering (*Lipatov '90s, Faddev, Korchemsky '90s*).
- Aim is to develop a systematic means to obtain classical continuum models from integrable spin chains that still preserve integrability! A non trivial task, this is the first systematic approach.

Outline

- Introduce the general underlying framework for classical and quantum integrable models.
- Introduce the long wave length limit. We provide a rigorous formulation associated to the whole hierarch of charges in involution.
- Examine particular examples. Start as a warm up exercise with the XXX model. Generalize our considerations to anisotropic models as well as to models associated to higher rank algebras.
- Discussion on possible future applications of the proposed methodology.

Algebraic setting

The essential building block the R matrix acting on $V_1 \otimes V_2$ is a solution of the Yang-Baxter equation (*Baxter '70s*)

$$R_{12}(\lambda_1 - \lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)$$

acting on $V_1 \otimes V_2 \otimes V_3$ and $R_{12} = R \otimes \mathbb{I},\ R_{23} = \mathbb{I} \otimes R$ etc. The classical limit:

$$R = 1 + \hbar r + \mathcal{O}(\hbar^2)$$

The classical r matrix satisfies the CYBE (*Semenov-Tian-Shansky '83*)

$$[r_{12},\ r_{13}] + [r_{12},\ r_{23}] + [r_{13},\ r_{23}] = 0.$$

Consider now the fundamental algebraic relation (seen as a generalization of the YBE) (*Faddev, Tahtajan, '80s*)

$$\boxed{R_{12}(\lambda_1 - \lambda_2)\ L_{1n}(\lambda_1)\ L_{2n}(\lambda_2) = L_{2n}(\lambda_2)\ L_{1n}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)}$$

Build tensor type representation of the algebra (quantum/deformed algebra) above (monodromy matrix):

$$T_a(\lambda) = L_{a1}(\lambda)\ L_{a2}(\lambda) \dots L_{aN}(\lambda)$$

T also satisfies the fundamental relation.

The classical limit may be then obtained

$$R = 1 + \hbar r + \ldots \quad \text{and set} \quad \frac{1}{\hbar}[\ ,\ ] = \{\ ,\ \}$$

then

$$\{L_a(\lambda),\ L_b(\lambda')\} = [r_{ab}(\lambda - \lambda'),\ L_a(\lambda) L_b(\lambda')]$$

Integrability

At both quantum and classical level one may construct charges in involution via:

$$t(\lambda) = tr_0[T_0(\lambda)]$$

Then via the fundamental algebraic relations:

$$[t(\lambda),\ t(\lambda')] = 0 \quad \text{quantum}; \qquad \{t(\lambda),\ t(\lambda')\} = 0 \quad \text{classical}$$

$\forall \lambda,\ \lambda'$. The latter guarantees the integrability of the system under consideration:

$$t(\lambda) = \sum \lambda^n I^{(n)}$$

$I^{(n)}$ the charges in involution:

$$[I^{(n)},\ I^{(m)}] = 0 \quad \text{quantum}; \qquad \{I^{(n)},\ I^{(m)}\} = 0 \quad \text{classical.}$$

The long wave-length limit

Start from local quantum discrete Hamiltonians (to obtain a local Hamiltonian $L(\lambda = \lambda_0) \propto \mathcal{P}$) (*Fradkin '91*):

$$H = \sum_{j=1}^{N} h_{jj+1}$$

Consider generic coherent state: $|n_j(x,t)\rangle$ such that

$$\boxed{h_j(x,t) = \langle n_j| \otimes \langle n_{j+1}| \ h_{jj+1} \ |n_j\rangle \otimes |n_{j+1}\rangle}$$

Consider the thermodynamic limit $N \to \infty$

$$l_j \to l(x,t), \qquad l_{j+1} \to l(x+\delta, t)$$

Also $\log t(\lambda) = \sum \lambda^n H^{(n)}$ ($H^{(m)}$ quantum local Hamiltonians) and

$$\boxed{\mathcal{H}^{(m)}(x,t) = \otimes_{j=1}^{N} \langle n_j| \ H^{(m)} \ \ \otimes_{j=1}^{N} |n_j\rangle}$$

So far the procedure applied only on local Hamiltonians without explicit proof of integrability. Here a systematic formulation guaranteeing integrability is proposed (*Avan, Doikou, Sfetsos, '10*). More precisely:

$$< T_a >= \langle n_1| \otimes \dots \langle n_j| \left(L_{a1}\ \dots L_{aN}\ |n_1\right)\rangle \otimes \dots |n_j\rangle \dots |n_1\rangle$$

then

$$T_a = \prod_{m=1}^{N} \langle n_m|\ L_{am}\ |n_m\rangle$$

but

$$< L_a >= 1 + \delta l_a + \delta^2 l_a^{(2)} + \dots$$

δ spacing parameter, $\delta \sim \frac{1}{N}$.

Then the monodromy matrix becomes:

$$T_a(\lambda) = 1 + \delta \sum_i l_{ai} + \delta^2 \sum_{i<j} l_{ai} l_{aj} + \delta^2 \sum_i l_{aj}^{(2)} + \dots$$

A subtle technical point: when taking the continuum limit of the expression above the *red* terms are *not* considered. Recall that $\delta \sim \frac{1}{N}$ and also

$$\sum f \rightarrow \frac{1}{N} \int f(x) dx$$

And the classical continuum limit gives the familiar monodromy:

$$T(\lambda) = P \exp\{ \int_0^A l(x) dx \}$$

To ensure integrability l is required (key point!) to satisfy

$$\boxed{\{l_a(\lambda, x),\ l_b(\lambda', y)\} = [r_{ab}(\lambda - \lambda'),\ l_a(\lambda, x) + l_b(\lambda, x)]\delta(x - y)}$$

ultra-local theory. And consequently:

$$\{T_a(\lambda), T_b(\lambda')\} = [r_{ab}(\lambda - \lambda'),\ T_a(\lambda) T_b(\lambda')].$$

Important proposition: let the quantum discrete local Hamiltonians obtained via (the same expression holds for classical integrals of motion)

$$H^{(n)} = \frac{d^n(\ln t(\lambda))}{d\lambda^n}\Big|_{\lambda=\lambda_0}$$

then one can prove (we skip the proof, quite technical see (*Avan, Doikou, Sfetsos, '10*))

$$< H^{(n)} > = < \frac{d^n \ln(t(\lambda))}{d\lambda^n} > \Big|_{\lambda=\lambda_0} = \frac{d^n(\ln < t(\lambda) >}{d\lambda^n}\Big|_{\lambda=\lambda_0}$$

The latter proposition together with the requirement that $< L >$ satisfies the quadratic classical algebra guarantees the integrability of the final classical models as well as the consistency of the whole process!

Specific examples

Consider the XXX models as a warm up exercise (see also *Fradkin '91* only for the long wave-length limit of the XXX Hamiltonian). The XXX Hamiltonian is:

$$H = \frac{1}{2}\sum_{j=1}^{N}\Big(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z\Big).$$

Introduce the coherent state

$$|n(x,t)\rangle = \cos\theta(x,t)e^{i\phi(x,t)}|+\rangle + \sin\theta(x,t)e^{-i\phi(x,t)}|-\rangle$$

and

$$\int d\mu(n)|n\rangle\langle n| = 1$$

with appropriate integration measure.

We want to compute the Hamiltonian density:

$$h = \langle n| \otimes \langle \tilde{n}|\ H_{jj+1}\ |n\rangle \otimes |\tilde{n}\rangle = |\langle n|\tilde{n}\rangle|^2.$$

Setting $\tilde{\theta} = \theta(x+\delta)$, $\ \tilde{\phi} = \phi(x+\delta)$ (keep the first non-trivial contribution δ^2):

$$h(x) = \theta'^2(x) + \sin^2(2\theta(x))\ \phi'^2(x).$$

The classical Hamiltonian then becomes

$$\boxed{\mathcal{H} \propto \int dx\ (\theta'^2(x) + \sin^2(2\theta(x))\ \phi'^2(x))}$$

In fact this is the so-called isotropic Landau-Lifshitz model (*Faddeev, Takhtajan '87*). Indeed parametrize:

$$S^z = \cos 2\theta \ , \qquad S^{\pm} = \frac{1}{2}\sin 2\theta \ e^{\mp 2i\phi} \ .$$

We obtain from the fundamental relation that

$$\{S^+, S^-\} = S^z \delta(x-y) \ , \qquad \{S^z, S^{\pm}\} = \pm 2 S^{\pm} \delta(x-y) \ .$$

The continuum parameters $\theta(x)$ and $\phi(x)$ also expressed in terms of canonical variables p and q:

$$\cos 2\theta(x) = p(x), \qquad \phi(x) = q(x), \qquad \{q(x),\ p(y)\} = i\delta(x-y) \ .$$

In terms of the classical spin variable the Hamiltonian is written

$$\boxed{H \propto \int dx \ \left(\left(\frac{dS^z}{dx} \right)^2 + \left(\frac{dS^x}{dx} \right)^2 + \left(\frac{dS^y}{dx} \right)^2 \right)} \ .$$

Starting from the underlying algebra will obtain in a straightforward manner the classical spin variables.

The R matrix (solution of the YBE) associated to the XXX model

$$R(\lambda) = 1 + \frac{i\hbar}{\lambda}\mathcal{P} = 1 + i\hbar \begin{pmatrix} \frac{1}{2}\sigma^z + \frac{1}{2} & \sigma^- \\ \sigma^+ & -\frac{1}{2}\sigma^z + \frac{1}{2} \end{pmatrix}$$

the classical r matrix is $r = \frac{1}{\lambda}\mathcal{P}$. Let $L(\lambda) = R(\lambda - \frac{i\hbar}{2})$ then

$$< L_{an}(\lambda) >= 1 + i\hbar l_{an}(\lambda)$$

with

$$l = \begin{pmatrix} \frac{1}{2}\langle n|\sigma^z|n\rangle & \langle n|\sigma^-|n\rangle \\ \langle n|\sigma^+|n\rangle & -\frac{1}{2}\langle n|\sigma^z|n\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2}S^z e & S^- \\ S^+ & -\frac{1}{2}S^z \end{pmatrix}.$$

The key point is: $< \sigma^i >= S^i$, and the basic requirement to ensure integrability is l to satisfy linear classical algebraic relations.

The anisotropic model

The XXZ Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{N}\Big(J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z\Big).$$

Set

$$J_\xi = 1 - \delta^2\alpha_\xi, \qquad \xi = x,\ y,\ z$$

Then recalling that: $\langle n|\ \sigma^i\ |n\rangle = S^i$ and following the long wave length procedure described above:

$$H \propto \int dx\ \left(\left(\frac{dS_z}{dx}\right)^2 + \left(\frac{dS_x}{dx}\right)^2 + \left(\frac{dS_y}{dx}\right)^2 + a_x S_x^2 + a_y S_y^2 + a_z S_z^2\right)$$

The anisotropic Landau-Lifshitz model, a *deformation* of the isotropic L-L model.

The classical r-matrix for the XXZ model is given as

$$r(\lambda) = \frac{1}{\sinh\lambda}\begin{pmatrix}(\frac{1}{2}\sigma^z + \frac{1}{2})\cosh\lambda & \sinh(i\mu)\sigma^- \\ \sinh(i\mu)\sigma^+ & (-\frac{1}{2}\sigma^z + \frac{1}{2})\cosh\lambda\end{pmatrix}$$

setting $L(\lambda) = R(\lambda - \frac{i\mu}{2})$ and

$$< L(\lambda) >= 1 + i\mu l(\lambda) + ...$$

where

$$l(\lambda) = \frac{1}{\sinh\lambda}\begin{pmatrix}\frac{1}{2}S^z\cosh\lambda & \sinh(i\mu)S^- \\ \sinh(i\mu)S^+ & -\frac{1}{2}S^z\cosh\lambda\end{pmatrix}$$

and recall l satisfies the underlying algebra defined via the linear relation

$$\{l_a(\lambda, x),\ l(\lambda', y)\} = [r_{ab}(\lambda - \lambda'),\ l_a(\lambda, x) + l_b(\lambda', y)]\delta(x - y).$$

Higher rank algebras

Expanding appropriately the local Hamiltonian, we conclude that

$$\mathcal{H}^{(0)}(x) = \sum_{i,j=1}^{n} l_{ij}(x)\ l_{ji}(x) - \frac{\delta^2}{2} \sum_{i,j=1}^{n} \frac{dl_{ij}(x)}{dx} \frac{dl_{ji}(x)}{dx}\ ,$$

The first term above is the quadratic Casimir and can be dropped. The term proportional to δ^2, provides the classical Hamiltonian of the generalized L-L model:

$$\boxed{H^{(0)} \propto \int dx\ \sum_{i,j=1}^{n} \frac{dl_{ij}(x)}{dx} \frac{dl_{ji}(x)}{dx}}\ .$$

Discussion

- Generalize the methodology to obtain for instance the sine Gordon (Liouville models) from the XXZ chain via the described process. Use of 'dualities' emerging within this context. Also discuss generalization i.e. $U_q(gl_n)$ spin chains to provide $A^{(1)}_{n-1}$ affine Toda theories.
- Super-symmetric generalization of the described methodology in order to further understand the integrable structures emerging in SYM theories.
- Study of non-ultra local quantum and classical sigma models. The classical integrable model within $\mathcal{N}=4$ SYM obeys non-ultra local algebras. Study of the quantum analogue is still an open question.