Introduction to quantum integrability

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Why Integrability?

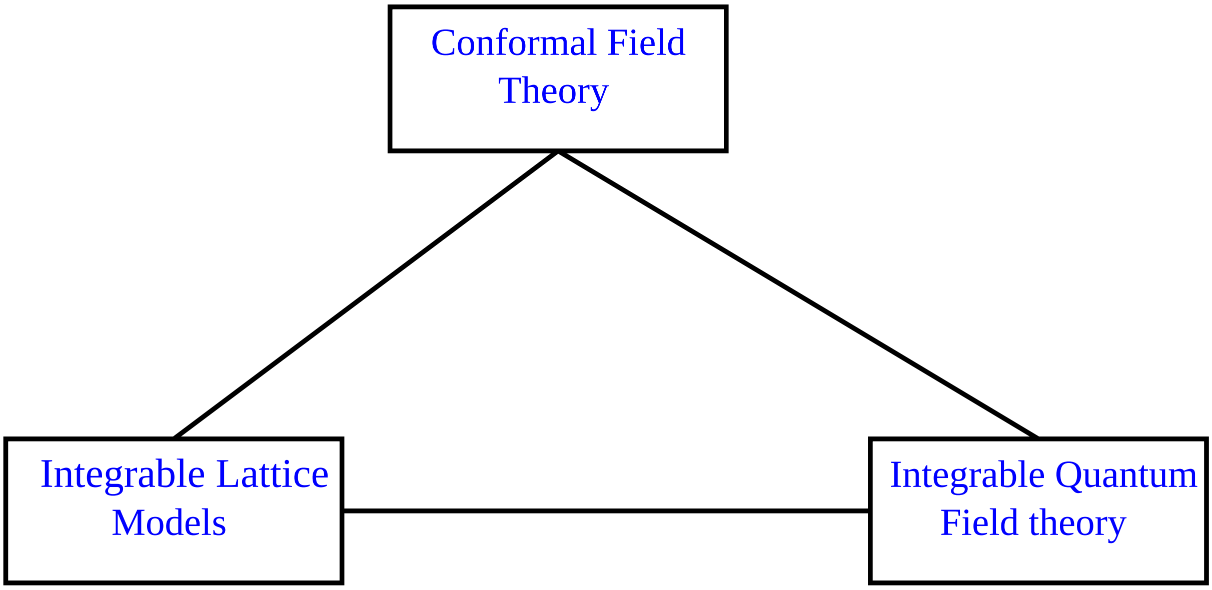
- Integrable Models: Exactly solvable models

- NON PERTURBATIVE methods: exact results!

- Interface, Mathematics–Physics. Wealth of applications and relations to other research areas

Relations/Applications

- Statistical Mechanics (*Onsager, Bethe, Baxter, McCoy*...)
- Condensed matter physics, e.g. Kondo effect, quantum Hall effect, disorder systems (*Affleck, Korepin, Saleur, Tsvelik, Wiegmann*....)
- High energy physics: QCD (*Lipatov, Faddeev, Korchemsky*), super YMT (*Minahan–Zarembo*...)
- String theory via CFT, D–branes via BCFT (*Polchinski*...)
- Mathematical aspects: quantum groups, braids, Lie and Hecke algebras, Virasoro algebras...(*Drinfeld, Faddeev, Jimbo, Kulish, Sklyanin, Reshetikhin*...)



Nice aspect!

• Perturbed CFT $\longrightarrow$ IQFT (*Zamolodchikov '89*)

• Critical statistical models (ILM) $\longrightarrow$ CFT (*Belavin, Polyakov, Zamolodchikov '84*)

• Light cone continuum limit of ILM $\longrightarrow$ IQFT (*Destri and de Vega '92*)

Algebraic Bethe ansatz

- Introduce the basic building block of the theory, R matrix $\longrightarrow$ transfer matrix ($Faddeev,\ Takhtajan,\ Sklyanin...$)

- **Main aim**: Diagonalization of the transfer matrix via the algebraic Bethe ansatz method $\longrightarrow$ BAE

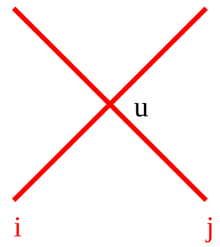
- Obtain quantities of physical interest: Energy momentum spectrum, quantum numbers, S–matrix, free energy, correlation functions...

- Quantum spin chains natural realizations of quantum algebras.

Study scattering of the low lying excitations:
exact S matrices

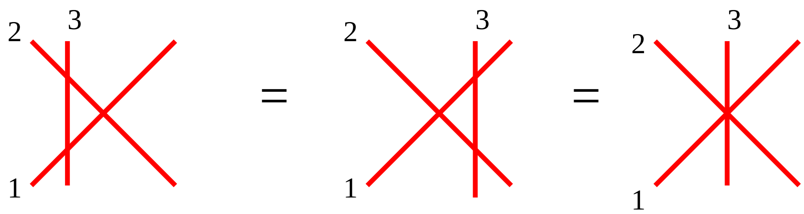
The R matrix

The R matrix acts on $\mathbb{V}^{\otimes 2}$:



$R_{ij}(u)$

Satisfies the YBE (*Baxter '72*)



$$R_{12}(\lambda_1 - \lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

- Physical interpretation of R: scattering among excitations
- YBE factorization condition of multiparticle scattering

History

- Many body (δ type) interaction (*Yang '67*). Bethe ansatz framework (*Gaudin '71*): YBE appears as factorization condition

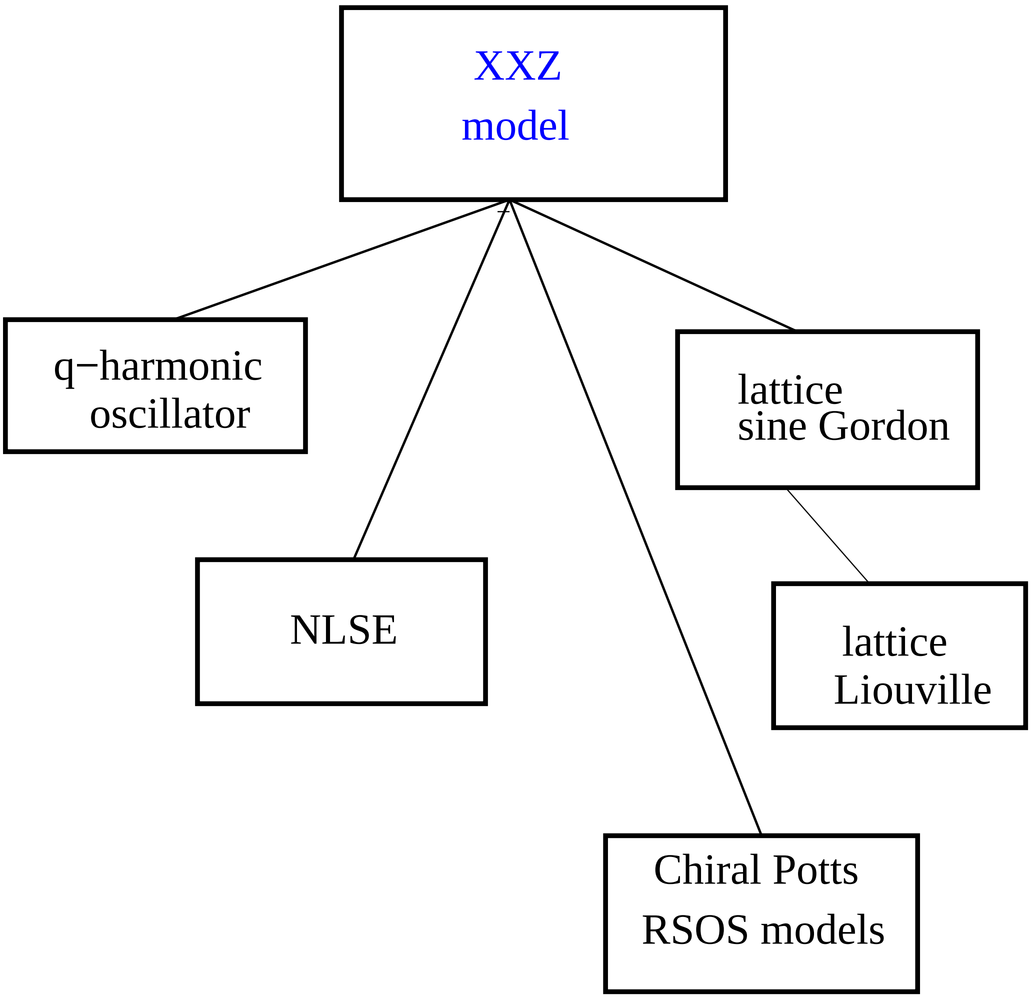
- Statistical Mechanics (**via YBE**) commuting transfer matrices (*Baxter '72*)

- Theory of factorized scattering "bootstrap" in relativistic setting (*Zamolodchikov, Berlin group, late 70's*)

- *Faddeev, Korepin, Kulish, Reshetikhin, Sklyanin, Takhtajan, ..., late 70's* introduced $QISM$ method: factorized scattering + soliton theory. Quantum algerbas arise naturally in this context.

The XXZ model

• XXZ model in particular 'Universal model' (*Faddeev, Izergin, Korepin...*) associated with many integrable models (quantum):



The Hamiltonian

$$H = -\frac{1}{4}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \cosh i\mu\sigma_j^z\sigma_{j+1}^z\right)$$

with periodic BC

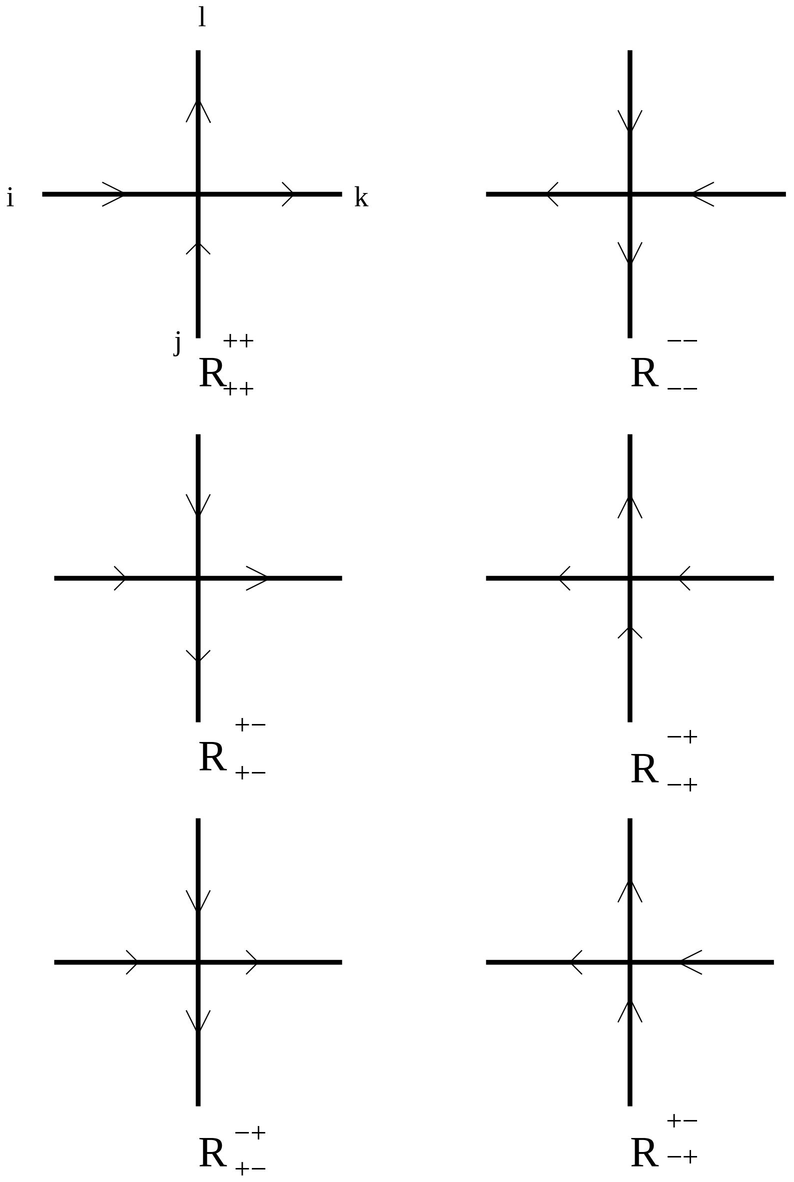
$$\sigma_1^i = \sigma_{1+N}^i$$

The corresponding R matrix acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$R(\lambda) = \begin{pmatrix} R_{++}^{++}(\lambda) & 0 & 0 & 0 \\ 0 & R_{+-}^{-+}(\lambda) & R_{+-}^{+-}(\lambda) & 0 \\ 0 & R_{-+}^{-+}(\lambda) & R_{-+}^{+-}(\lambda) & 0 \\ 0 & 0 & 0 & R_{--}^{--}(\lambda) \end{pmatrix}$$

The 6-vertex model

'Ice rule': $i + j = k + l$



Relax constraint 8-vertex: $i + j = k + l \ mod(2)$

R matrices from the Hecke algebra

Let $\check{R}_{12}(\lambda) = \mathcal{P}_{12}\, R_{12}(\lambda)$, $\quad \mathcal{P}(x \otimes y) = y \otimes x$, then YBE:

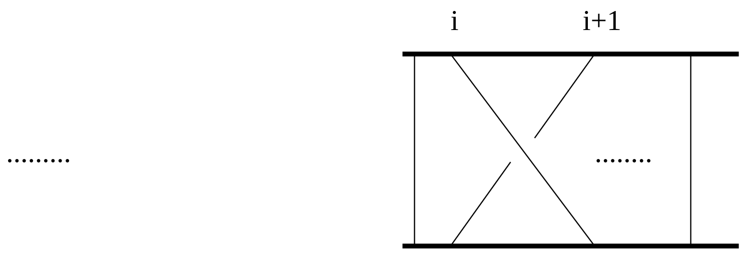
$$\check{R}_{12}(\lambda_1 - \lambda_2)\; \check{R}_{23}(\lambda_1)\; \check{R}_{12}(\lambda_2) = \check{R}_{23}(\lambda_2)\; \check{R}_{12}(\lambda_1)\; \check{R}_{23}(\lambda_1 - \lambda_2).$$

The Hecke algebra $\mathcal{H}_N(q)$ with g_i, $i = 1, \ldots, N-1$:

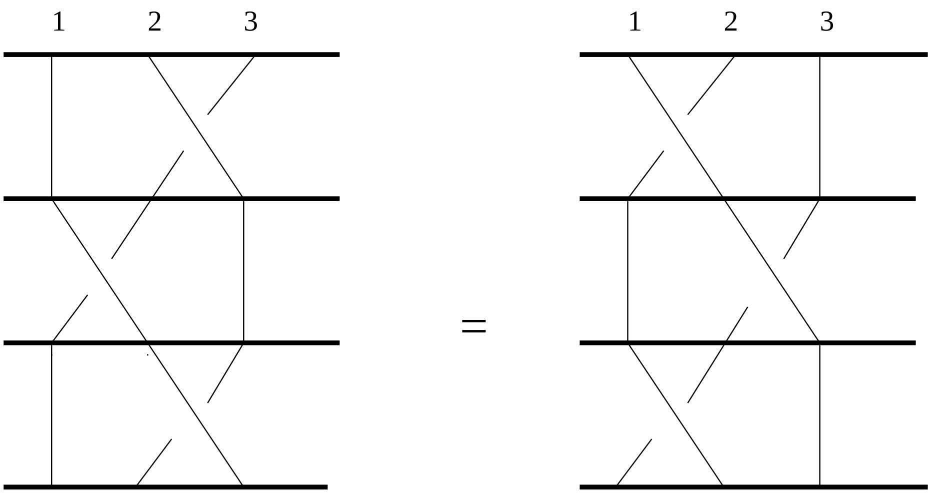
$$(g_i - q)\; (g_i + q^{-1}) = 0$$
$$g_i\; g_{i+1}\; g_i = g_{i+1}\; g_i\; g_{i+1},$$
$$[g_i,\; g_j] = 0, \quad |i - j| > 1.$$

The structural similarity between the *braid* relation and YBE:
$\mathcal{H}_N \longrightarrow$ candidate solutions of YBE.

Braid graphical representation



g_i



g2 g1 g2 g1 g2 g1

Let $e_i = g_i - q$ then reps of $\mathcal{H}_N \longrightarrow$ solution YBE (Baxterization) (*Jimbo '86*)

$$\check{R}_{i\ i+1}(\lambda) = \sinh[\mu(\lambda+i)]\ \mathbb{I} + \sinh(\mu\lambda)\ e_i$$
$$= e^{\mu\lambda} g_i - e^{-\mu\lambda} g_i^{-1}.$$

In particular, let $(E_{ij})_{kl} = \delta_{ik}\ \delta_{jl}$, and U on $(\mathbb{C}^n)^{\otimes 2}$:

$$U = \sum_{i \neq j} (E_{ij} \otimes E_{ji} - q^{-sgn(i-j)} E_{ii} \otimes E_{jj}).$$

provides the R-matrix in the fundamental representation of $U_q(gl_n)$.

Generalize this method for the boundary YBE (reflection) equation *Levy, Martin '94, Doikou, Martin, '02, '05*.

$U_q(sl_2)$: The XXZ model

U on $(\mathbb{C}^2)^{\otimes 2}$

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding R (spin $\frac{1}{2}$ rep of sl_2) matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$ (rep of Temperley–Lieb algebra)

$$R(\lambda) = \begin{pmatrix} \sinh\mu(\lambda+i) & 0 & 0 & 0 \\ 0 & \sinh\mu\lambda & e^{\mu\lambda}\sinh i\mu & 0 \\ 0 & e^{-\mu\lambda}\sinh i\mu & \sinh\mu\lambda & 0 \\ 0 & 0 & 0 & \sinh\mu(\lambda+i) \end{pmatrix}$$

Temperley-Lieb algebra

U provides representation of the Temperley-Lieb algebra $\mathbb{T}_N$ (quotient of $\mathcal{H}_N$) defined by $e_i,\ i \in \{1, \dots N-1\}$ and relations:

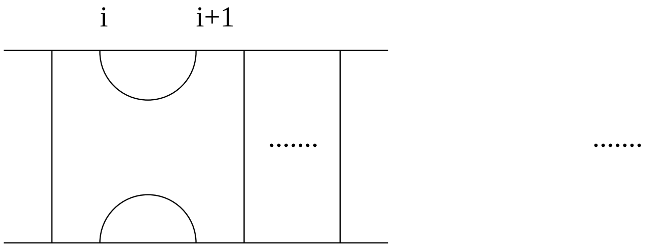
$$\begin{aligned}
&e_{j\pm1}\ e_j\ e_{j\pm1} = e_{j\pm1} \\
&e_i^2 = -(q + q^{-1})e_i \\
&[e_i,\ e_j] = 0, \quad |i - j| > 1
\end{aligned}$$

Let $\pi : \mathbb{T}_N \longrightarrow \mathrm{End}((\mathbb{C}^2)^{\otimes N})$ with:

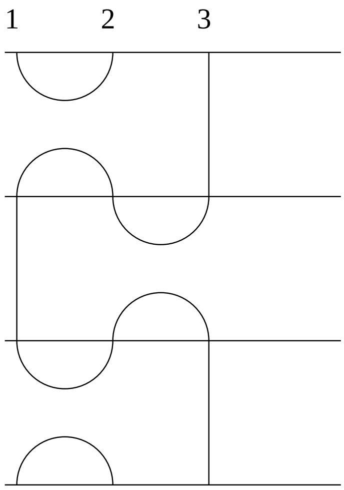
$$U_i = \pi(e_i)$$

$$U_i = \mathbb{I} \otimes \dots\ \mathbb{I} \otimes \underbrace{U}_{i,\ i+1} \otimes \mathbb{I} \dots \otimes \mathbb{I}$$

Graphical representation of TL algebra

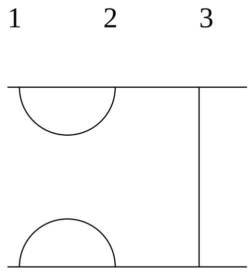


ei

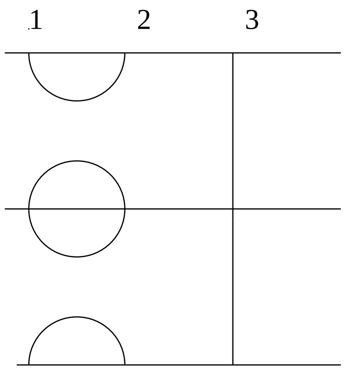


e1 e2 e1

=

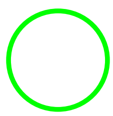


e1

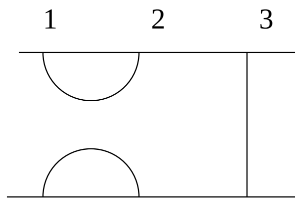


e1 e1

=



−(q + 1/q)



e1

The Lax operator

Rewrite the R matrix in terms of Pauli matrices:

$$R(\lambda) = \begin{pmatrix} e^{\mu\lambda} q^{\frac{\sigma^z}{2}} - e^{-\mu\lambda} q^{-\frac{\sigma^z}{2}} & (q - q^{-1}) e^{\mu\lambda} \sigma^- \\ (q - q^{-1}) e^{-\mu\lambda} \sigma^+ & e^{\mu\lambda} q^{-\frac{\sigma^z}{2}} - e^{-\mu\lambda} q^{\frac{\sigma^z}{2}} \end{pmatrix}$$

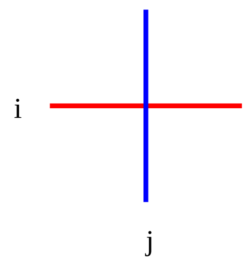
For any representation of $U_q(sl_2)$: $\mathcal{L}$ matrix,

$$\mathcal{L}(\lambda) = \begin{pmatrix} e^{\mu\lambda}\mathrm{A} - e^{-\mu\lambda}\mathrm{D} & (q - q^{-1}) e^{\mu\lambda}\mathrm{B} \\ (q - q^{-1}) e^{-\mu\lambda}\mathrm{C} & e^{\mu\lambda}\mathrm{D} - e^{-\mu\lambda}\mathrm{A} \end{pmatrix}$$

$A,\ B,\ C,\ D$ generate $U_q(sl_2)$:

$$\mathrm{A}\ \mathrm{D} = \mathrm{D}\ \mathrm{A} = \mathbb{I}, \qquad \mathrm{A}\ \mathrm{C} = q\mathrm{C}\ \mathrm{A}, \qquad \mathrm{A}\ \mathrm{B} = q^{-1}\mathrm{B}\ \mathrm{A},$$
$$\left[\mathrm{C},\ \mathrm{B}\right] = \frac{\mathrm{A}^2 - \mathrm{D}^2}{q - q^{-1}}.$$

The $\mathcal{L}$ matrix acts on $\mathbb{V} \otimes \mathcal{A}\ (U_q(\widehat{sl_2}))$, $\ q = e^{i\mu}$:



Satisfies the defining relation of $\mathcal{A}$

$$R_{12}(\lambda_1 - \lambda_2)\ \mathcal{L}_{13}(\lambda_1)\ \mathcal{L}_{23}(\lambda_2) = \mathcal{L}_{23}(\lambda_2)\ \mathcal{L}_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

Heisenberg-Weyl group

Express A, B, C, D in terms of the Heisenberg-Weyl group:

$$X\ Z = q\ Z\ X$$

then

$$\mathrm{A} = \mathrm{D}^{-1} = X, \quad \mathrm{B} = \frac{1}{q-q^{-1}}(q^{-s}X^{-1} - q^{s}X)Z^{-1},$$
$$\mathrm{C} = \frac{1}{q-q^{-1}}(q^{-s}X - q^{s}X^{-1})Z$$

For $q^d = 1$ admits d dim. 'cyclic' representation (*Roche, Arnaudon '89*).
In addition cyclicity:

$$X^d = Z^d = \mathbb{I}.$$

Use: Hofstadter Hamiltonian (Bloch electrons in a magnetic field: Azbel-Hofstadter problem) (*Wiegmann, Zabrodin '93, Faddeev, Kashaev '93, Floratos, Nicolis '95*).
Also quantum Hall effect states as cyclic reps of $U_q(sl_2)$.
Open boundary conditions (*Doikou '06*).

For

$$X = e^{-i\Phi}, \qquad Z = e^{i\Pi}, \qquad \Big[\Phi_n,\ \Pi_m\Big] = i\mu\delta_{mn}.$$

obtain the lattice sine-Gordon (*Izergin, Korepin '81*). Limits $\rightarrow$ Lattice Liouville, q-harmonic oscillator...

$$L^{SG}_{0n}(\lambda) = \begin{pmatrix} h_+(\Phi_n)e^{i\Pi_n} & -im\ \sinh(\lambda + i\Phi_n) \\ -im\ \sinh(\lambda - i\Phi_n) & h_-(\Phi_n)e^{-i\Pi_n} \end{pmatrix}$$

$$h_\pm(\Phi_n) = 1 + m^2 e^{\pm 2i\Phi_n + i\mu}$$

$(m \rightarrow 0)$ (*Faddeev, Tirkkonen '95*) $\rightarrow$ the lattice Liouville model:

$$L^{Lv}_{0n}(\lambda) = \begin{pmatrix} e^{i\Pi_n} & -ie^{-\lambda - i\Phi_n} \\ -i\sinh(\lambda - i\Phi_n) & h(\Phi_n)e^{-i\Pi_n} \end{pmatrix}$$

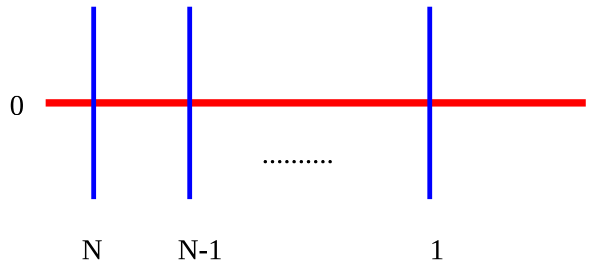
$$h(\Phi_n) = 1 - e^{-2i\Phi_n + i\mu}$$

Similar limiting process leads to the q-harmonic oscillator L matrix. (Open boundary conditions (*Doikou '06*)).

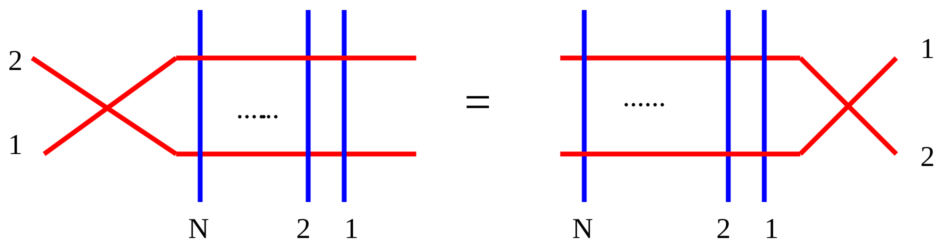
Tensor representations: the periodic spin chain

The monodromy matrix $T \in \mathrm{End}(\mathbb{V}) \otimes \mathcal{A}^{\otimes(N)}$ (QISM: *Faddeev, Takhtajan '81*):

$$T_0(\lambda) = \mathcal{L}_{0N}(\lambda)\ \mathcal{L}_{0\ N-1}(\lambda) \ldots \mathcal{L}_{01}(\lambda)$$



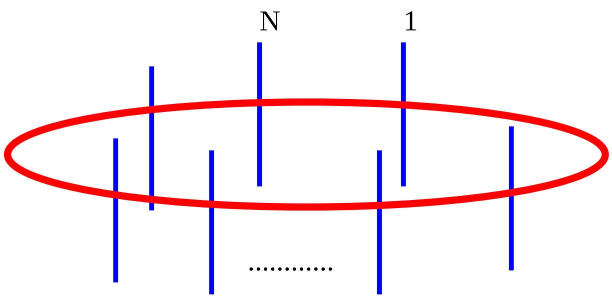
Satisfies the fundamental algebraic relation:



$$R_{12}(\lambda_1 - \lambda_2)\ T_1(\lambda_1)\ T_2(\lambda_2) = T_2(\lambda_2)\ T_1(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

The transfer matrix $t \in \mathcal{A}^{\otimes N}$ (*Faddeev, Takhtajan '81*):

$$t(\lambda) = Tr_0 \; \{T_0(\lambda)\}$$



Provides a family of commuting operators

$$\Big[t(\lambda), \; t(\lambda') \Big] = 0$$

Latter commutation relation ensures Integrability

$$\mathcal{H} \propto \frac{d}{d\lambda} (ln \; t(\lambda))|_{\lambda=0}$$

Quantum algebras: deformed coproduct

$$\mathcal{L}(\lambda) = e^{\mu\lambda}\mathcal{L}^{+} - e^{-\mu\lambda}\mathcal{L}^{-}$$

$\mathcal{L}^{\pm}_{ab} \in U_q(gl_n)$

As $\lambda \to \pm\infty$ $\mathcal{L}$ and consequently T reduce to upper, lower triangular matrices.

$$T(\lambda \to \pm\infty) \propto T^{\pm},$$

entries of $T^{\pm} \in U_q(gl_n)^{\otimes N}$

e.g. $U_q(sl_2)$

$$\mathcal{L}(\lambda) = e^{\mu\lambda}\begin{pmatrix} q^{s^z} & s^{-}2\sinh i\mu \\ 0 & q^{-s^z} \end{pmatrix} - e^{-\mu\lambda}\begin{pmatrix} q^{-s^z} & 0 \\ -s^{+}2\sinh i\mu & q^{s^z} \end{pmatrix}$$

Then asymptotics of T:

$$T^+ \propto \begin{pmatrix} q^{S^z} & c^+ S^- \\ 0 & q^{-S^z} \end{pmatrix}, \quad T^- \propto \begin{pmatrix} q^{-S^z} & 0 \\ c^- S^+ & q^{S^z} \end{pmatrix}$$

$$S^z = \sum_{k=1}^{N} \mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes s_k^z \otimes \mathbb{I} \ldots \otimes \mathbb{I},$$
$$S^\pm = \sum_{k=1}^{N} q^{-s_1^z} \otimes \ldots \otimes q^{-s_{k-1}^z} \otimes s_k^\pm \otimes q^{s_{k+1}^z} \otimes \ldots \otimes q^{s_N^z}$$

S^z, $S^\pm$ tensor product realizations of $U_q(sl_2)$ (*Jimbo '85*)

$$[S^+,\ S^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}}, \qquad [S^z,\ S^\pm] = \pm S^\pm$$

Study of the underlying quantum algerbas (Yangians): $so(n)$, $sp(m)$, $osp(n|m)$, $sl(n|m)$: *Arnaudon, Avan, Crampe, Doikou, Frappat, Ragoucy, '03–'05*

Bethe ansatz

• Main aim: diagonalization of transfer matrix via algebraic Bethe ansatz.

1. Reference state: highest weight
2. Use the RTT algebra exchange relations.

• Find the spectrum, analyticity requir. provide BAE.

• BAE important, their solution $\rightarrow$ physically relevant quantities: exact S matrices, thermodynamic properties, correlation functions...

Diagonalization of $t(\lambda)$

The $\mathcal{L}$–matrix rewritten as

$$\mathcal{L}_{0n}(\lambda) = \begin{pmatrix} \alpha_n^+ & \beta_n \\ \gamma_n & \alpha_n^- \end{pmatrix}$$

$$\alpha_n^\pm = \sinh \mu(\lambda \pm i s_n^z), \quad \gamma_n = s_n^+ \sinh i\mu, \quad \beta_n = s_n^- \sinh i\mu$$

Reference state, highest weight:

$$\gamma_n |+\rangle_n = 0, \qquad |\Omega\rangle = \bigotimes_{n=1}^{N} |+\rangle_n \,.$$

and consequently

$$T(\lambda)|\Omega\rangle = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ 0 & \mathcal{D}(\lambda) \end{pmatrix} |\Omega\rangle$$

the diagonal entries of T acting on the pseudovacuum give,

$$\mathcal{A}(\lambda)|\Omega\rangle = \sinh^N \mu(\lambda + is)|\Omega\rangle, \;\; \mathcal{D}(\lambda)|\Omega\rangle = \sinh^N \mu(\lambda - is)|\Omega\rangle$$

Assumption the general Bethe state has the form

$$|\psi\rangle = \mathcal{B}(\lambda_1)\,\mathcal{B}(\lambda_2)\ldots\mathcal{B}(\lambda_M)\,|\Omega\rangle$$

Solve the eigenvalue problem,

$$t(\lambda)|\psi\rangle = \Big(\mathcal{A}(\lambda) + \mathcal{D}(\lambda)\Big)|\psi\rangle = \Lambda(\lambda)|\psi\rangle$$

Commutation relations: $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$, $\mathcal{B}$, from $RTT = TTR$, With the help of $TTR = RTT$ obtain the eigenvalues of $t(\lambda)$

$$\Lambda(\lambda) = \prod_{j=1}^{M} \frac{\sinh \mu(\lambda - \lambda_j - i)}{\sinh \mu(\lambda - \lambda_j)} \sinh^N \mu(\lambda + is) + \prod_{j=1}^{M} \frac{\sinh \mu(\lambda - \lambda_j + i)}{\sinh \mu(\lambda - \lambda_j)} \sinh^N \mu(\lambda - is)$$

The analyticity of $\Lambda \to \lambda$'s satisfy BAE

$$e_{2s}(\lambda_i) = \prod_{j=1}^{M} e_2(\lambda_i - \lambda_j)$$

where $e_n(\lambda) = \frac{\sinh \mu(\lambda + \frac{in}{2})}{\sinh \mu(\lambda - \frac{in}{2})}$

BAE $\to$ physical quantities : S-matrix, free energy, specific heat, central charge...

Energy momentum and spin in terms of BA roots

Energy

$$E = -\frac{1}{2\pi} \sum_{j=1}^{M} \frac{\mu \sinh i\mu}{\sinh \mu(\lambda_j + \frac{i}{2}) \sinh \mu(\lambda_j - \frac{i}{2})}$$

Momentum

$$P = -\sum_{j=1}^{M} i \ln \frac{\sinh \mu(\lambda_j + \frac{i}{2})}{\sinh \mu(\lambda_j - \frac{i}{2})}$$

Spin

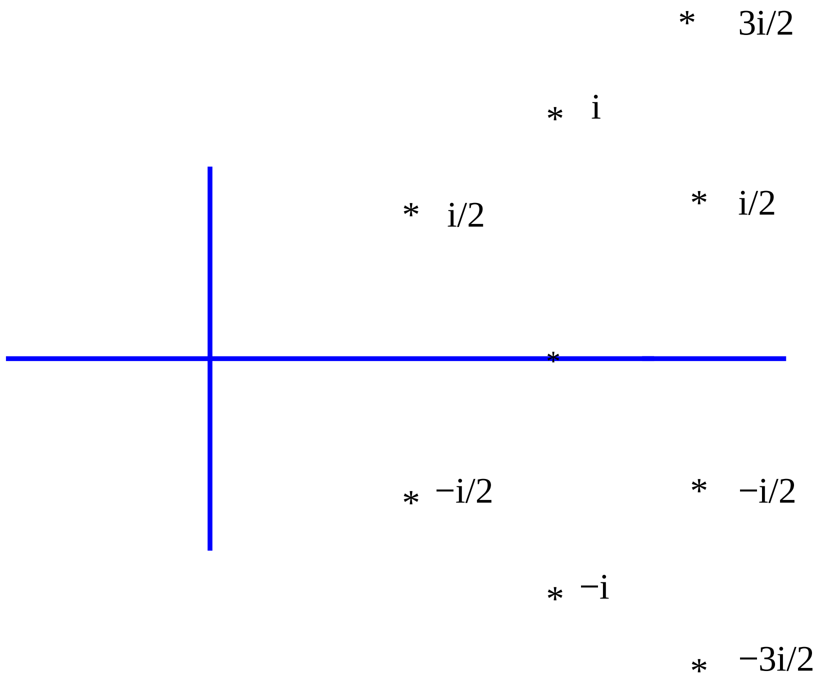
$$S^z = \frac{N}{2} - M$$

Note: $E = \frac{1}{2\pi} \frac{dP}{d\lambda}$

String hypothesis

Solutions of BAE for $N \to \infty$ may be casted as (*Faddeev and Takhtajan '81*):

$$\lambda^{(n,j)} = \lambda_0^{(n)} + \frac{i}{2}(n + 1 - 2j)$$



Solving Bethe ansatz equations

Ground state: all real strings, filled Dirac sea $S^z = 0$.

Low lying excitations: Holes in the filled Dirac sea, particle like excitations

- Spin $S^z = \frac{1}{2}$
- Energy $\epsilon(\lambda) = \frac{1}{2\cosh\frac{\pi\lambda}{2}}$

I hole $\longrightarrow$ 2D rep of $SU(2)$. State with 2 holes, the density (from the BAE):

$$\sigma(\lambda) = 2\pi\epsilon(\lambda) + \frac{1}{N} r(\lambda)$$

$\hat{r}(\omega) = \frac{\sinh(\frac{\pi}{\mu}-2)\frac{\omega}{2}}{2\cosh\frac{\omega}{2}\sinh(\frac{\pi}{\mu}-1)\frac{\omega}{2}}$.

• **Main aim**: derive 2-hole scattering amplitude
Quantization condition (*Korepin '79, Andrei and Destri '84*)

$$(e^{ipN}S - 1)|\lambda_i\rangle = 0$$

recall $\epsilon(\lambda) = \frac{1}{2\pi}\frac{dp(\lambda)}{d\lambda}$, compare the **QC** with the density:

$$S_0(\lambda) = exp\Big[-\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\hat{r}(\omega)e^{-i\omega\lambda}\Big]$$

$$S_0(\lambda) = exp\Big[-\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\frac{\sinh(\frac{\pi}{\mu}-2)\frac{\omega}{2}}{2\cosh\frac{\omega}{2}\sinh(\frac{\pi}{\mu}-1)\frac{\omega}{2}}e^{-i\omega\lambda}\Big]$$

sine Gordon S–matrix for $\beta^2 = 8(\pi-\mu)$ (*Zamolodchikov '79*).

More eigenvalues

$$S \propto R$$

We can find all the eigenvalues of the 4×4 S matrix: suitable string configurations

- **2 holes and a 2-string in the middle**

$$S_a(\lambda) = \frac{\sinh \mu(\lambda - i)}{\sinh \mu(\lambda + i)} S_0(\lambda)$$

- **2 holes and a negative parity string in the middle**

$$S_b(\lambda) = \frac{\cosh \mu(\lambda - i)}{\cosh \mu(\lambda + i)} S_0(\lambda)$$

Method applied for higher rank algebras, super–algebras: *Doikou and Nepomechie '97–'99, Doikou '00, Arnaudon, Avan, Crampe, Doikou, Frappat, Ragoucy, '03–'05*