Systematic construction of (boundary) Lax pairs

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Motivation

- Integrable b.c. interesting for integrable systems per ce, new info on boundary phenomena + learn more on bulk behavior. Examples of integrable b.c. that modify the bulk.
- *Bigger picture*, strong motivations nowadays: recent developments within the AdS/CFT context (*Minahan, Zarembo '03*). Important to study both quantum and classical integrable models.
- Further: recent results on open spin chains and open string theories (*Agarwal, Hofman, Maldacena,...*)

Outline

- Brief overview on quantum integrability (mathematical and physical description). Review periodic and open boundary conditions.
- Classical discrete integrable models: Review general setting. Lax pair and algebraic description. Generalize the “boundary” case. Rigorous universal results on IM and Lax pairs based on the underlying algebra. Examples: Discrete-self-trapping (DST) model (discrete NLS).
- Classical continuum integrable models: similar investigations; IM and boundary Lax pairs from the algebraic setting. Examples: NLS and sine-Gordon models.

Review quantum integrability

- The Yang-Baxter equation (*Baxter '72*)

  $$R_{12}(\lambda_1-\lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2)=R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1-\lambda_2)$$

  R acts on $V\otimes V$, YBE on $V\otimes V\otimes V$, and $R_{12}=R\otimes\mathbb{I},\ R_{23}=\mathbb{I}\otimes R$ etc.
- R physically describes scattering, YBE the factorization of multi-particle scattering.
- V associated to reps of underlying (deformed) Lie algebras.

- Generalize the YBE to include generic reps of (deformed) Lie algebras (*Faddeev, Takhtajan Reshetikhin*):

$$R_{12}(\lambda_1-\lambda_2)\ L_{1n}(\lambda_1)\ L_{2n}(\lambda_2) = L_{2n}(\lambda_2)\ L_{1n}(\lambda_1)\ R_{12}(\lambda_1-\lambda_2)$$

  $L \in \text{End}(V \otimes \mathcal{A})$, $\mathcal{A}$ defined by fundamental algebraic relation above: *deformed or quantum algebras*.

- This describes physically integrable models with periodic bc.

Periodic integrable models

- Tensorial reps of the fundamental algebraic relation (*Faddeev, Takhtajan 80's*):

$$T_0(\lambda) = L_{0N}(\lambda)\ L_{0N-1}(\lambda)\dots L_{01}(\lambda)$$

- $T \in \mathrm{End}(V \otimes \mathcal{A}^{\otimes N})$ satisfies the fundamental algebraic relation. The trace over the "auxiliary space", defines the transfer matrix :

$$t(\lambda) = tr_0\, T_0(\lambda)$$

- Via the RTT relations integrability is shown:

$$[t(\lambda),\ t(\lambda')] = 0, \quad \forall\lambda,\ \lambda'$$

Open integrable models

- Introduce the reflection or boundary YBE (*Cherednik, Sklyanin '80s*)

$$R_{12}(\lambda_1 - \lambda_2)K_1(\lambda_1)R_{21}(\lambda_1 + \lambda_2)K_2(\lambda_2) =$$

$$K_2(\lambda_2)R_{12}(\lambda_1 + \lambda_2)K_1(\lambda_1)R_{12}(\lambda_1 - \lambda_2)$$

  $K \in \mathrm{End}(V)$, the reflection matrix.

- Physically describes the reflection of a particle-like excitation with the boundary of the system.

- Tensor reps of reflection equation (*Sklyanin '83*)

$$\mathbb{T}_0(\lambda) = T_0(\lambda)\ K_0(\lambda)\ T_0^{-1}(-\lambda)$$

- Define the open transfer matrix

$$t(\lambda) = tr_0\{K_0^+(\lambda)\ \mathbb{T}_0(\lambda)\}$$

- Using the reflection equation we show integrability

$$[t(\lambda),\ t(\lambda')] = 0, \quad \forall\lambda,\ \lambda'$$

Generating function of IM.

Classical limit

- Now focus on classical models. Consider the classical limit of the R matrix as:

$$R = 1 + \hbar r + \mathcal{O}(\hbar^2)$$

Then the r matrix satisfies the classical YBE (*Semenov-Tian-Shansky '83*)

$$[r_{12},\ r_{13}] + [r_{12},\ r_{23}] + [r_{13},\ r_{23}] = 0.$$

- The classical limit obtained by setting: $\frac{1}{\hbar}[\ ,\ ] = \{\ ,\ \}$

$$\{L_a(\lambda),\ L_b(\lambda')\} = [r_{ab}(\lambda - \lambda'),\ L_a(\lambda)L_b(\lambda')]$$

Systematic study of classical limit (*Avan, Doikou, Sfetsos '10*).

Discrete integrable classical models: periodic b.c.

- Lax pair $(L,\ A)$ for discrete integrable models, and the associated auxiliary problem

$$\psi_{n+1} = L_n\ \psi_n$$

$$\dot{\psi}_n = A_n\ \psi_n$$

- From the latter equations one obtains the discrete zero curvature condition:

$$\dot{L}_n = A_{n+1}\ L_n - L_n\ A_n$$

- L-operator and T satisfy the quadratic classical algebraic relation (*Faddeev, Takhtajan '87*):

$$\{L_a(\lambda),\ L_b(\lambda')\} = [r_{ab}(\lambda-\lambda'),\ L_a(\lambda)L_b(\lambda')]$$

- We formulate:

$$\begin{aligned}\{T_a(\lambda),\ L_{bn}(\lambda')\} &= T_a(N,n+1;\lambda)r_{ab}(\lambda-\lambda')T_a(n,1;\lambda)L_{bn}(\lambda') \\ &- L_{bn}(\lambda')T_a(N,n;\lambda)r_{ab}(\lambda-\lambda')T_a(n-1,1;\lambda)\end{aligned}$$

where $T_a(m,n;\lambda) = L_{am}(\lambda)\dots L_{an}(\lambda)$.

- Take the trace over the auxiliary space and the log then:

$$\{\ln t(\lambda),\ L(\lambda')\} = t^{-1} tr_a\Big(T_a(N, n+1;\lambda) r_{ab}(\lambda^-) T_a(n,1;\lambda)\Big) L_{bn}$$
$$-t^{-1} L_{bn}(\lambda') tr_a\Big(T_a(N,n;\lambda) r_{ab}(\lambda^-) T_a(n-1,1;\lambda)\Big)$$

$\lambda^- = \lambda - \lambda'$

- t is the generating function of all I.M.: $\ln t(\lambda)$ gives rise to all local Hamiltonians i.e.

$$\ln t(\lambda) = \sum_n \frac{\mathcal{H}^{(n)}}{\lambda^n}$$

- The time evolution of L is given as:

$$\{\ln t(\lambda),\ L(\lambda')\} = \dot{L}(\lambda')$$

- The Lax pair obtained comparing with the zero curvature condition as: L_n and

$$A_n(\lambda) = t^{-1}(\lambda) tr_a\Big(T_a(N,n;\lambda) r_{ab}(\lambda-\lambda') T_a(n-1,1;\lambda)\Big)$$

expansion in powers of λ provides all A's associated to all I.M.

- Expansion of $\ln t$ provides all local Hamiltonians.

$$A_n(\lambda) = \sum_n \frac{A_n^{(m)}}{\lambda^m}$$

One to one correspondence between Lax pairs and I.M.

Discrete integrable classical models: open b.c.

- The underlying algebra:

$$\begin{aligned}\{\mathbb{T}_a(\lambda),\ \mathbb{T}_b(\lambda')\} &= r_{ab}(\lambda^-)\mathbb{T}_a(\lambda)\mathbb{T}_b(\lambda') - \mathbb{T}_a(\lambda)\mathbb{T}_b(\lambda')r_{ba}(\lambda^-) \\ &+ \mathbb{T}_a(\lambda)r_{ba}(\lambda^+)\mathbb{T}_b(\lambda') - \mathbb{T}_b(\lambda')r_{ab}(\lambda^+)\mathbb{T}_a(\lambda)\end{aligned}$$

$\lambda^{\pm} = \lambda \pm \lambda'$.

- Recall the tensorial rep of the algebra

$$\mathbb{T}_a(\lambda) = T_a(\lambda)\ K_a(\lambda)\ T^{-1}(-\lambda)$$

K a c-number rep of the reflection algebra.

- Extract the associated boundary Lax pair (*Avan, Doikou '07*). Formulate

$$\{T_a(\lambda),\ L_{bn}(\lambda')\} = \dots$$
$$\{T_a^{-1}(-\lambda),\ L_{bn}(\lambda')\} = \dots$$

- Recall the generic rep of the reflection algebra and show that:

$$\{t(\lambda),\ L_{bn}(\lambda')\} = \dots$$

- We read of the boundary quantity $\mathbb{A}_n$, which satisfies the zero curvature condition together with L:

$$\mathbb{A}_n = tr_a\Big(K_a^+(\lambda)\,T_a(N,n;\lambda)r_{ab}(\lambda^-)\,T_a(n-1,1;\lambda)K_a^-(\lambda)\,\hat{T}_a(\lambda)$$
$$+K_a^+(\lambda)\,T_a(\lambda)K_a^-(\lambda)\,\hat{T}_a(1,n-1;\lambda)r_{ba}(\lambda^+)\,\hat{T}_a(n,N;\lambda)\Big)$$

- Special care at the boundary points $n=1,\ n=N$, recall $T(N,N+1)=T(0,1)=\hat{T}(1,0)=\hat{T}(N+1,N)=1$

Examples

- Focus next to simple models associated to the sl_2-Yangian. The classical r matrix is:

$$r(\lambda - \lambda') = \frac{\mathcal{P}}{\lambda - \lambda'}$$

$\mathcal{P}$ is the permutation operator $\mathcal{P}\ (a \otimes b) = b \otimes a$.

- Focus on the Discrete-Self-Trapping (DST) model

$$L(\lambda) = (\lambda - xX)\ e_{11} + b\ e_{22} + b\ x\ e_{12} - X\ e_{21}$$

$(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$, and

$$\{x_n,\ X_m\} = \delta_{nm}$$

- Expand the $t(\lambda)$ and keep the first charge, which is the Hamiltonian of the model. Focus on the simplest case: $K^{\pm} \propto \mathbb{I}$ then:

$$\mathcal{H}^{(2)} = -\frac{1}{2} \sum_{n=1}^{N} x_n^2 X_n^2 - b \sum_{n=1}^{N-1} x_{n+1} X_n - \frac{b^2}{2} x_1^2 - \frac{1}{2} X_N^2$$

- The associated equation of motion:

$$\dot{L} = \{\mathcal{H},\ L\}$$

- The expansion of $\mathbb{A}_n$ will provide the relevant boundary lax operator (*Avan, Doikou '07*):

$$\mathbb{A}_n{}^{(2)} = \begin{pmatrix} \lambda & bx_n \\ -X_{n-1} & 0 \end{pmatrix}, \quad n \in \{2, \dots, N\}$$

$$\mathbb{A}_1{}^{(2)} = \begin{pmatrix} \lambda & bx_1 \\ -bx_1 & 0 \end{pmatrix}, \quad \mathbb{A}^{(2)}_{N-1} = \begin{pmatrix} \lambda & X_N \\ -X_N & 0 \end{pmatrix}$$

- The associated equations of motion given as

$$\dot{L}_n = \mathbb{A}^{(i)}_{n+1}\, L_n - L_n \mathbb{A}^{(i)}_n$$

- For the specific example the relevant equations of motion are:

$$\dot{x}_n = x_n^2 X_n + b x_{n+1}, \qquad \dot{X}_n = -x_n X_n^2, \qquad n \in \{2, \dots, N-1\}$$
$$\dot{x}_1 = x_1^2 X_1 + b x_2, \qquad \dot{X}_1 = -x_1 X_1^2 - b x_1$$
$$\dot{x}_N = x_N^2 X_N + X_N, \qquad \dot{X}_N = -x_N$$

- The Toda chain also obtained for the DST model.
  Specifically, set:

$$X_n \to e^{-q_n}, \qquad x_n \to e^{q_n}(b^{-1} + p_n)$$

- The harmonic oscillator algebra $(x_n,\ X_n,\ x_n X_n)$ reduces to the Euclidean Lie algebra $(e^{\pm q_n},\ p_n)$ and the Lax operator:

$$L(\lambda) = \begin{pmatrix} \lambda - p_n & e^{q_n} \\ -e^{-q_n} & 0 \end{pmatrix}$$

- The Hamiltonian is then:

$$\mathcal{H}^{(2)} = -\frac{1}{2}\sum_{i=1}^{N} p_n^2 - \sum_{n=1}^{N-1} e^{q_{n+1}-q_n} - \frac{1}{2}e^{2q_1} - \frac{1}{2}e^{-2q_N}$$

- The relevant equations of motion for the open Toda chain:

$$p_n = \dot{q}_n, \qquad \ddot{q}_n = e^{q_{n+1}-q_n} - e_{q_n - q_{n-1}}, \qquad n \in \{2, \dots, N-1\}$$
$$p_1 = q_1, \qquad \ddot{q}_1 = e^{q_2 - q_1} - e^{2q_1}$$
$$p_N = q_N, \qquad \ddot{q}_N = e^{-2q_N} - e^{q_N - q_{N-1}}$$

Continuum integrable classical models on the full line

- Let $\mathbb{U}$, $\mathbb{V}$ be the continuum Lax pair, and Ψ be the solution of the following set of equations:

$$\frac{\partial \Psi}{\partial x} = \mathbb{U}(x, t, \lambda)\ \Psi$$
$$\frac{\partial \Psi}{\partial t} = \mathbb{V}(x, t, \lambda)\ \Psi$$

- Compatibility condition of the above set gives the zero curvature condition

$$\dot{\mathbb{U}} - \mathbb{V}' + [\mathbb{U},\ \mathbb{V}] = 0$$

- Solution of the 1st equ. (monodromy):

$$T(x, y; \lambda) = \mathrm{P}\exp\{\int_y^x \mathbb{U}(x', t, \lambda) dx'\}$$

- T satisfies the fundamental quadratic relation. Formulate (*Faddeev, Takhtajan '87*):

$$\{T_a(L, -L, \lambda),\ \mathbb{U}(x,\ \lambda)\} = \frac{\partial M(x, \lambda, \lambda')}{\partial x} + [M(x, \lambda, \lambda'), \mathbb{U}_b(x,\ \lambda)]$$

where

$$M = T_a(L, x, \lambda)\ r_{ab}(\lambda - \lambda')\ T_a(x, -L, \lambda)$$

- It then follows:

$$\{\ln t(\lambda),\ \mathbb{U}(x, \lambda)\} = \frac{\partial \mathbb{V}}{\partial x} + [\mathbb{V}(x, \lambda, \lambda'),\ \mathbb{U}(x, \lambda)]$$

$\mathbb{V}$ identified as:

$$\mathbb{V}(x, \lambda, \lambda') = t^{-1}(\lambda) tr_a(M_{ba})$$

Continuum integrable classical models on the interval

- Recalling that the rep of the reflection algebra is

$$\mathbb{T}_a(x, y, \lambda) = T_a(x, y, \lambda)\ K_a(\lambda)\ \hat{T}_a(x, y, \lambda)$$

- Formulating $\{T,\ \mathbb{U}\},\ \{\hat{T},\ \mathbb{U}\}$ (*Avan, Doikou '07*):

$$\{\ln t(\lambda),\ \mathbb{U}(x, \lambda')\} = \frac{\partial \mathbb{V}(x, \lambda, \lambda')}{\partial x} + [\mathbb{V}(x, \lambda, \lambda'),\ \mathbb{U}(x, \lambda, \lambda')]$$

and

$$\mathbb{V}(x, \lambda, \lambda') = t^{-1}(\lambda) tr_a \Big(K_a^+(\lambda)\ \mathbb{M}_a(x, \lambda, \lambda') \Big)$$

- The boundary quantity $\mathbb{M}$ is

$$\begin{aligned}\mathbb{M} &= T(0,x,\lambda)r_{ab}(\lambda-\lambda')T(x,-L,\lambda)K^{-}(\lambda)\hat{T}(0,-L,\lambda)) \\ &+ T(0,-L,\lambda)K^{-}(\lambda)\hat{T}(x,-L,\lambda)r_{ba}(\lambda+\lambda')\hat{T}(0,x,\lambda)\end{aligned}$$

pay particular attention at the boundary points (key point!) $x=0,\ -L$; take into account $T(x,x,\lambda)=\hat{T}(x,x,\lambda)=\mathbb{I}$.

- *Systematic derivation independent of the choice of model, as opposed to 'by hand' construction in earlier investigations e.g. ATFT (Bowcock, Corrigan, Dorey, Rietdijk '95)*

Example: boundary NLS model

- Consider the NLS model. The associated r matrix

$$r(\lambda) = \frac{\mathcal{P}}{\lambda}$$

- Recall the Lax pair

$$\mathbb{U} = \mathbb{U}_0 + \lambda \mathbb{U}_1, \qquad \mathbb{V} = \mathbb{V}_0 + \lambda \mathbb{V}_1 + \lambda^2 \mathbb{V}_2$$

where

$$\mathbb{U}_1 = \frac{1}{2i}(e_{11} - e_{22}), \qquad \mathbb{U}_0 = \bar{\psi} e_{12} + \psi e_{21}$$

$$\mathbb{V}_0 = i|\psi|^2 (e_{11} - e_{22}) - i\bar{\psi}' e_{12} + i\psi' e_{21}$$

$$\mathbb{V}_1 = -\mathbb{U}_0, \qquad \mathbb{V}_2 = -\mathbb{U}_1$$

- The fields $\psi,\ \bar{\psi}$ are canonical

$$\{\psi(x),\ \bar{\psi}(y)\} = \delta(x-y)$$

- From the zero curvature conditions the equations of motion for NLS:

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{\partial^2\psi(x,t)}{\partial^2 x} + 2|\psi(x,t)|^2\psi(x,t)$$

- Idea: via the process described integrable boundary conditions (system on the half line).

- Choose diagonal reflection matrix:

$$K(\lambda) = \lambda(e_{22} - e_{11}) + i\xi\mathbb{I}$$

- Consider the ansatz:

$$T(x, y, \lambda) = \Big(1 + W(x, \lambda)\Big)\ e^{Z(x,y,\lambda)}\ \Big(1 + W(y, \lambda)\Big)^{-1}$$

Z is purely diagonal, W off diagonal. Determine Z, W from $\frac{d}{dxT} = \mathbb{U}T$, and find the associated integrals of motion.

- Expanding the transfer matrix obtain the classical integrals of motion for NLS in the interval (*Doikou, Fioravanti, Ravanini, '07*):

$$\mathcal{N} = \int_{-L}^{0} dx\ \psi(x)\bar{\psi}(x)$$

$$\mathcal{H} = \int_{-L}^{0} dx\ \Big(|\psi(x)|^4 + \psi'(x)\bar{\psi}'(x)\Big)$$
$$-\psi(0)\bar{\psi}'(0) - \psi'(0)\bar{\psi}(0) - \xi^+\psi(0)\bar{\psi}(0)$$
$$+\psi(-L)\bar{\psi}'(-L) + \psi'(-L)\bar{\psi}(-L) + \xi^-\psi(-L)\bar{\psi}(-L)$$

- *Only odd charges are conserved!* E.g. momentum is not a conserved quantity anymore.

- The corresponding equations of motion obtained from:

$$\frac{\partial \psi(x,t)}{\partial x} = \{\mathcal{H},\ \psi(x,t)\}, \quad \frac{\partial \bar{\psi}(x,t)}{\partial x} = \{\mathcal{H},\ \bar{\psi}(x,t)\}$$
$$-L \leq x \leq 0$$

- And are of the form:

$$i\frac{\partial \psi(x,t)}{\partial t} = -\frac{\partial^2 \psi(x,t)}{\partial^2 x} + 2|\psi(x,t)|^2 \psi(x,t)$$
$$\Big(\frac{\partial \psi(x)}{\partial x} - \xi^{\pm}\psi(x)\Big)_{x=0,\ -L} = 0.$$

Mixed boundary conditions.

- Following the generic formulation described we identify the associated boundary Lax pair (*Avan, Doikou '07*):

$$\mathbb{V}_b(x_b, t) = \mathbb{V}(x_b, t) + \Delta\mathbb{V}(x_b, t)$$
$$= \mathbb{V}(x_b, t) + i|\psi(x_b, t)|^2 e_{22} + \lambda\Big(\bar{\psi}(x_b, t)e_{12} + \psi(x_b, t)e_{21}\Big)$$

$x_b = 0, \ -L.$

- Obtain the equations of motion, and continuity arguments at the boundary point lead to $\Delta\mathbb{V} = 0 \rightarrow$ boundary conditions

Example: boundary sine-Gordon

- The Lax pair for the sine Gordon model ($u = e^{\lambda}$)

$$\mathbb{U}(x,t,u) = \frac{\beta}{4i}\pi(x) + \frac{mu}{4i} e^{\frac{i\beta}{4}\phi\sigma_3}\sigma_2 e^{-\frac{i\beta}{4}\phi\sigma_3} - \frac{mu^{-1}}{4i} e^{-\frac{i\beta}{4}\phi\sigma_3}\sigma_2 e^{\frac{i\beta}{4}\phi\sigma_3}$$

$$\mathbb{V}(x,t,u) = \frac{\beta}{4i}\phi'(x) + \frac{mu}{4i} e^{\frac{i\beta}{4}\phi\sigma_3}\sigma_2 e^{-\frac{i\beta}{4}\phi\sigma_3} + \frac{mu^{-1}}{4i} e^{-\frac{i\beta}{4}\phi\sigma_3}\sigma_2 e^{\frac{i\beta}{4}\phi\sigma_3}$$

- The classical r-matrix for the sine-Gordon model (*Jimbo '86*)

$$r(\lambda) = \frac{\cosh\lambda}{\sinh\lambda}\sum_{i=1}^{2} e_{ii}\otimes e_{ii} + \frac{1}{\sinh\lambda}\sum_{i\neq j=1}^{2} e_{ij}\otimes e_{ji}$$

- Choose the reflection matrix (*Ghoshal, Zamolodchikov '94*)

$$K(\lambda) = \begin{pmatrix} \sinh(\lambda + i\xi) & x^{+}\kappa\sinh(2\lambda) \\ x^{-}\kappa\sinh(2\lambda) & \sinh(-\lambda + i\xi) \end{pmatrix}, \qquad x^{-}\ x^{+} = 1$$

- The boundary Hamiltonian via the process described obtained (*MacIntyre '95*), also at quantum level (*Ghoshal, Zamolodchikov '94*):

$$\mathcal{H} = \int_{-L}^{0} dx \left(\frac{\beta}{4i}(\pi^2 + \phi'^2) + \frac{m^2}{\beta^2}(1 - \cos\beta\phi)\right) + \frac{4Pm}{\beta^2}\cos\frac{\beta\phi(0)}{2} - \frac{4Qm}{\beta^2}\sin\frac{\beta\phi(0)}{2}$$

$$P = -\frac{\sin\xi}{2i\kappa}, \qquad Q = \frac{\cos\xi}{2i\kappa}$$

- Generalize to ATFT's, full classification of integrable b.c., classical I.M. (*Doikou '08*)

- The boundary Lax pair reads (*Avan, Doikou '08*):

$$\mathbb{V}^{(b)}(0,t,u) = \mathbb{V}(0,t,u) + \Delta\mathbb{V}(0,t,u)$$
$$\Delta\mathbb{V}(0,t,u) = -\frac{\beta}{4i}\phi'(0)\sigma_3 - \frac{m}{8\kappa}\cos(\xi + \frac{\beta}{2}\phi(0))\sigma_3$$

  Different choices of reflection matrices modify the boundary conditions: Dirichlet, Neumann or mixed

- Generalize the construction of boundary Lax pairs to ATFT's (*Avan, Doikou '08*)

- From the zero curvature condition and continuity requirement $\Delta \mathbb{V} = 0$ we get the E.M and the boundary conditions:

$$\ddot{\phi}(x,t) - \phi''(x,t) = -\frac{m^2}{\beta} \sin(\beta\phi(x,t))$$

$$\beta\phi'(0) = \frac{m}{2i\kappa} \cos(\xi + \frac{\beta}{2}\phi(0))$$

Mixed boundary conditions.

Discussion

- Results have been obtained for models associated to higher rank Lie algebras e.g. vector NLS model (*Doikou, Fioravanti, Ravanini '07*) and ATFT (*Doikou '08*, and *Avan, Doikou '08*).

- Full classification of integrable boundary conditions in these models. Two distinct types of boundary conditions emerge: the soliton preserving, and the soliton non-preserving. *Dynamical boundaries*, e.g. coupled harmonic oscillator at the ends.

- Full classification of integrable boundary conditions at the quantum level as well (*Doikou '00*, and *Arnaudon, Avan, Crampe, Doikou, Frappat, Ragoucy '03, '04*).