ΠΑΝΕΠΙΣΤΗΜΙΟ ΠΑΤΡΩΝ
ΠΟΛΥΤΕΧΝΙΚΗ ΣΧΟΛΗ, ΓΕΝΙΚΟ ΤΜΗΜΑ

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PHYSICAL AND ALGEBRAIC ASPECTS OF QUANTUM INTEGABILITY

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Why Integrability?

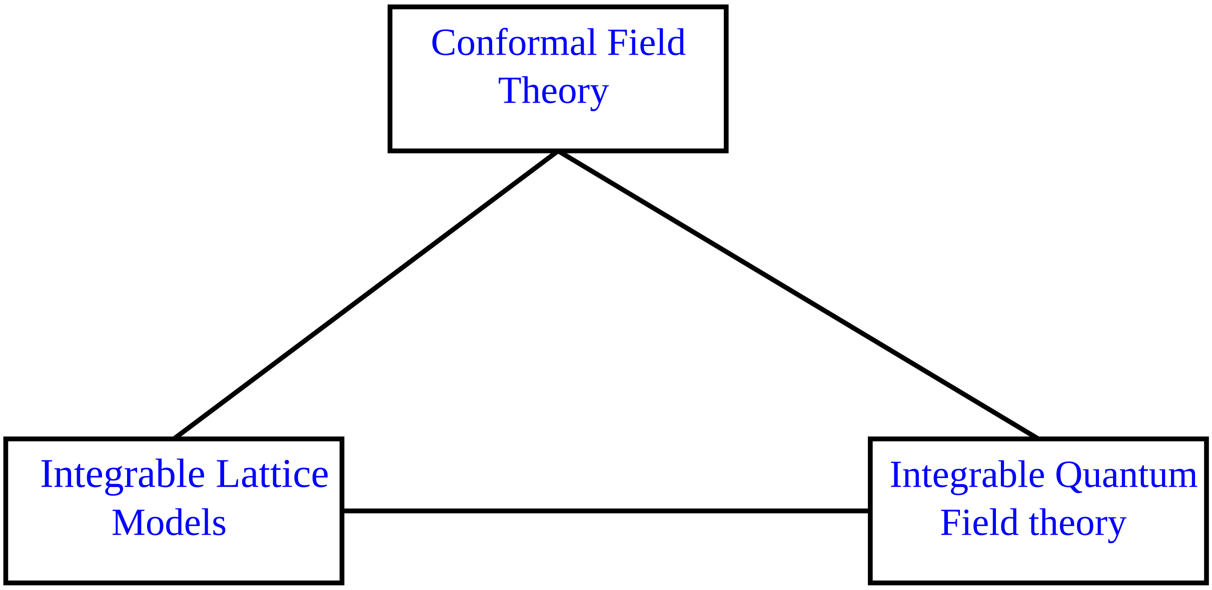
- Integrable Models: Exactly solvable models

- NON PERTURBATIVE methods: exact results!

- Interface, Mathematics–Physics. Wealth of applications and relations to other research areas

Relations/Applications

- Statistical Mechanics (*Onsager, Bethe, Baxter, McCoy*...)
- Condensed matter physics, e.g. Kondo effect, quantum Hall effect, disorder systems (*Affleck, Korepin, Saleur, Tsvelik, Wiegmann*....)
- High energy physics: QCD (*Lipatov, Faddeev, Korchemsky*), super YMT (*Minahan–Zarembo*...)
- String theory via CFT, D–branes via BCFT (*Polchinski*...)
- Mathematical aspects: quantum groups, braids, Lie and Hecke algebras, Virasoro algebras...(*Drinfeld, Faddeev, Jimbo, Kulish, Sklyanin, Reshetikhin*...)



Nice aspect!

- Perturbed CFT $\longrightarrow$ IQFT (*Zamolodchikov '89*)

- Critical statistical models (ILM) $\longrightarrow$ CFT (*Belavin, Polyakov, Zamolodchikov '84*)

- Light cone continuum limit of ILM $\longrightarrow$ IQFT (*Destri and de Vega '92*)

History

- Heisenberg model solved (*Bethe '31*). Factorization of multi-particle interaction $\rightarrow$ 2-particle interaction!!

- Many body (δ type) interaction (1D boss-gas (*Lieb-Lininger '67*), N interacting fermions: (*Yang '67*). Bethe ansatz framework (*Gaudin '71*): YBE appears as factorization condition

- Statistical Mechanics (**via YBE**) commuting transfer matrices (*Baxter '72*). Ising model solved many years ago (*Onsager '40s*) also integrable model.

- Theory of factorized scattering "bootstrap" in relativistic setting (*Zamolodchikov, Berlin group, late 70's*)

- *Faddeev, Korepin, Kulish, Reshetikhin, Sklyanin, Takhtajan, ..., late 70's* introduced $QISM$ method: factorized scattering + soliton theory. Quantum algerbas arise naturally in this context.

The XXZ Hamiltonian

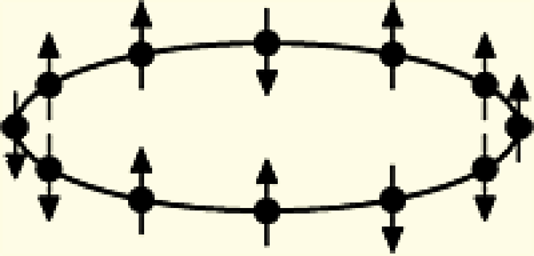
$$H = -\frac{1}{4}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \cosh(i\mu)\ \sigma_j^z\sigma_{j+1}^z\right)$$

with periodic BC

$$\sigma_1^i = \sigma_{1+N}^i.$$

The Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



σ_i provide the spin-$\frac{1}{2}$ representation of $su(2)$ ($s_1,\ s_2,\ s_3$):

$$[s_i,\ s_j] = 2i\epsilon_{ijk}\ s_k$$

and

$$s_1 \hookrightarrow \sigma_1 \qquad s_2 \hookrightarrow \sigma_2, \qquad s_3 \hookrightarrow \sigma_3$$

Also define $s^{\pm} = \frac{1}{2}(s_1 \pm i s_2)$ creation-annihilation operators as in the Harmonic operator. Alternatively $su(2)$:

$$[s^+,\ s^-] = s_3, \qquad [s^{\pm},\ s_3] = \pm s^{\pm}$$

Bethe's solution

Wish to solve a typical eigenvalue problem (diagonalize the Hamiltonian):

$$H \ |\Psi >= \Lambda \ |\Psi >$$

H in general operator in terms of abstract s_i. Now is represented to spin $\frac{1}{2}$ $\longrightarrow \ 2^N$ matrix!

Diagonalize a $2^N \times 2^N$ matrix...one has to be smart!!

Bethe ansatz

Start with a state (ferromagnetic vacuum) all spins up. Let n-spins being down. These are called pseudo-particles. The state with n spin down $|x_1, \ x_2, \ \ldots x_n >$.

Bethe worked in the configuration space of pseudo-particles. Parametrize the sate as:

$$|\Psi >= \sum_{1 \leq x_1 \leq x_2 ... x_n \leq N} a(x_1, x_2, ..., x_n)|x_1,\ x_2,\ \ldots, x_n >$$

Bethe's great insight (ansatz)!! Deduced $a(x_1, x_2, ..., x_n)$ as:

$$a(x_1, x_2, ..., x_n) = \sum_{P \in S_n} A_P\ \exp[ik_{p_i} x_i]$$

$P = (p_1, p_2, ..., p_n) \in S_n$ and k_i the momentum of the pseudo-particle at x_i.

Obtain A_P in terms of 2-particle interaction!!! Already see a nice structure.

$$A_P = \epsilon_P \prod_{1 \le i < j \le n} S_{p_i p_j}$$

S_{ij} depends on the momenta k_i, scattering of 2 pseudo-particle. A_P scattering of n pseudo-particles

k_i satisfy the Bethe ansatz equations

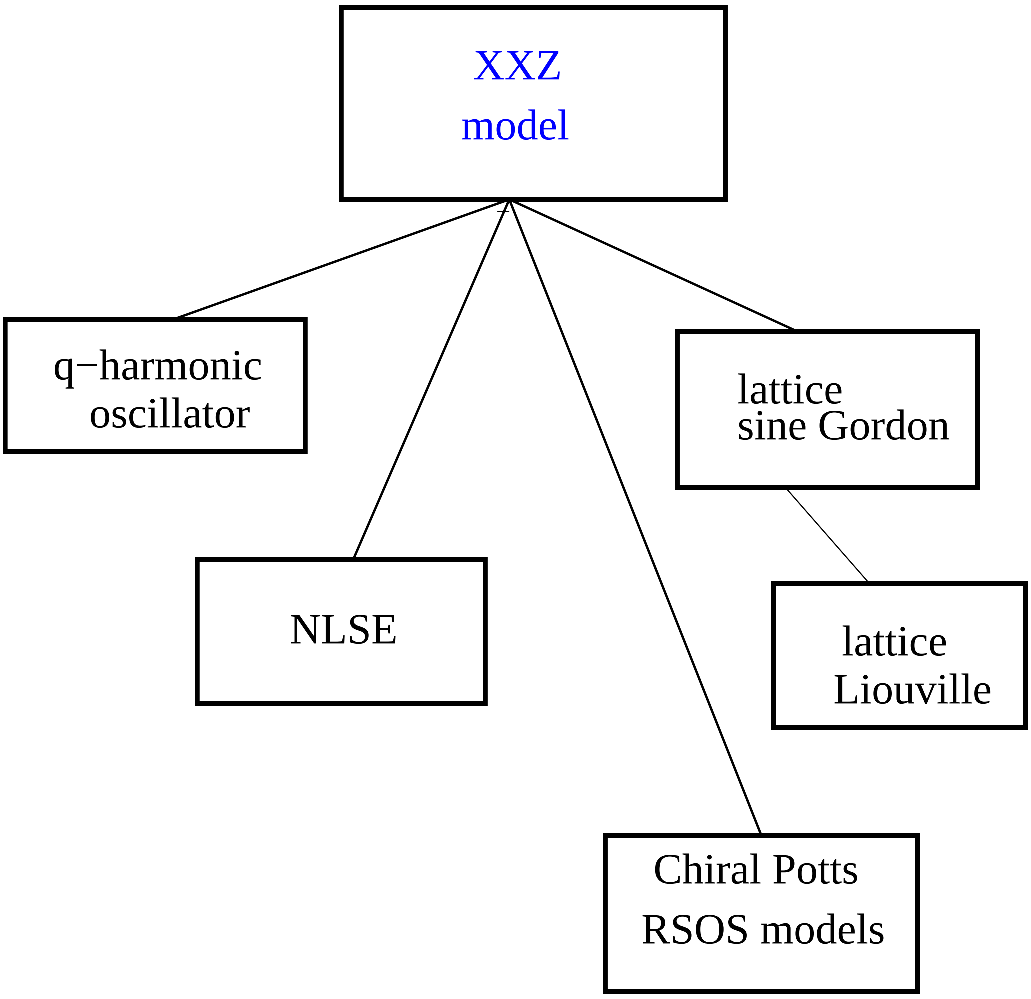
$$\exp[i k_j N] = (-)^{n-1} \prod_{j \neq l} \frac{S_{lj}}{S_{jl}}$$

FACTORIZATION OF MULTI-PARTICLE INTERACTION!!

Unique feature of quantum integrable models

The XXZ model

• XXZ model in particular 'Universal model' (*Faddeev, Izergin, Korepin...*) associated with many integrable models (quantum):



Algebraic Bethe ansatz

- Introduce the basic building block of the theory, R matrix $\longrightarrow$ transfer matrix ($Faddeev,\ Takhtajan,\ Sklyanin...$)

- **Main aim**: Diagonalization of the transfer matrix via the algebraic Bethe ansatz method $\longrightarrow$ BAE

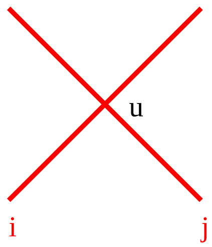
- Obtain quantities of physical interest: Energy momentum spectrum, quantum numbers, S–matrix, free energy, correlation functions...

- Quantum spin chains natural realizations of quantum algebras.

Study scattering of the low lying excitations:
exact S matrices

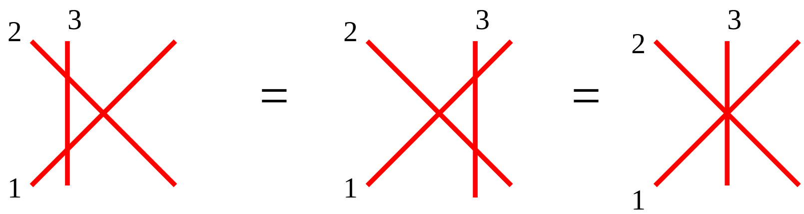
The R matrix

The R matrix acts on $\mathbb{V}^{\otimes 2}$:



$R_{ij}(u)$

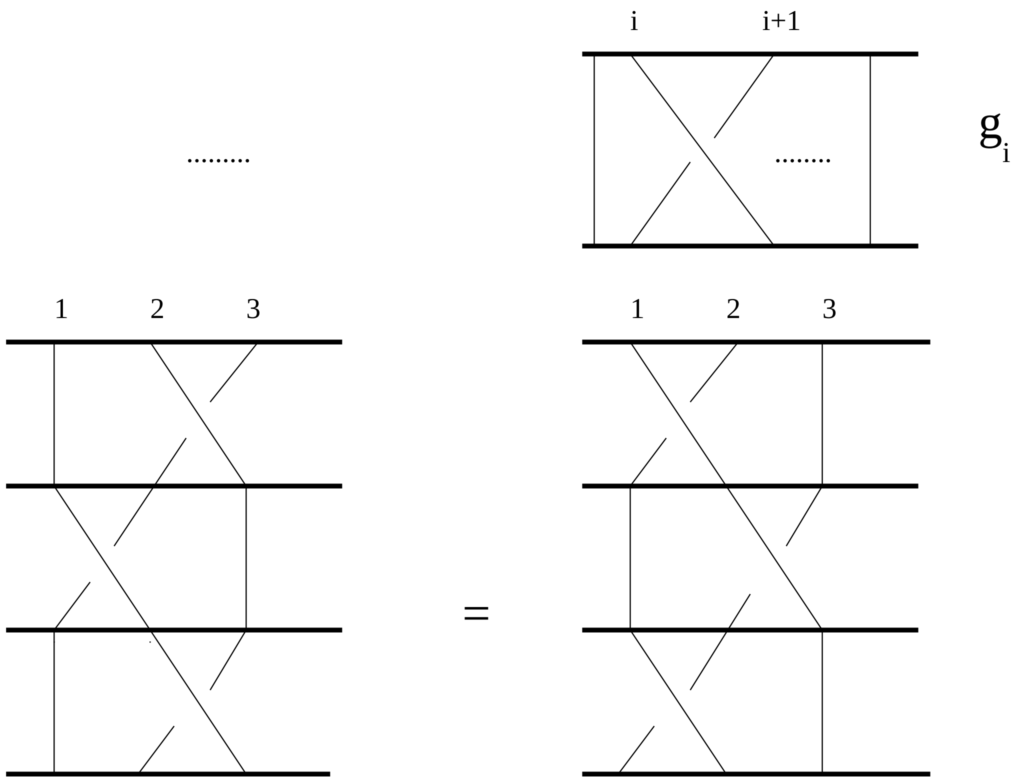
Satisfies the YBE (*Baxter '72*)



$$R_{12}(\lambda_1 - \lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

- Physical interpretation of R: scattering among excitations
- YBE factorization condition of multiparticle scattering

Braid graphical representation



g2 g1 g2

g1 g2 g1

The XXZ R-matrix acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$, solution of the Yang-Baxter equation:

$$R(\lambda) = \begin{pmatrix} R_{++}^{++}(\lambda) & 0 & 0 & 0 \\ 0 & R_{+-}^{-+}(\lambda) & R_{+-}^{+-}(\lambda) & 0 \\ 0 & R_{-+}^{-+}(\lambda) & R_{-+}^{+-}(\lambda) & 0 \\ 0 & 0 & 0 & R_{--}^{--}(\lambda) \end{pmatrix}$$

where

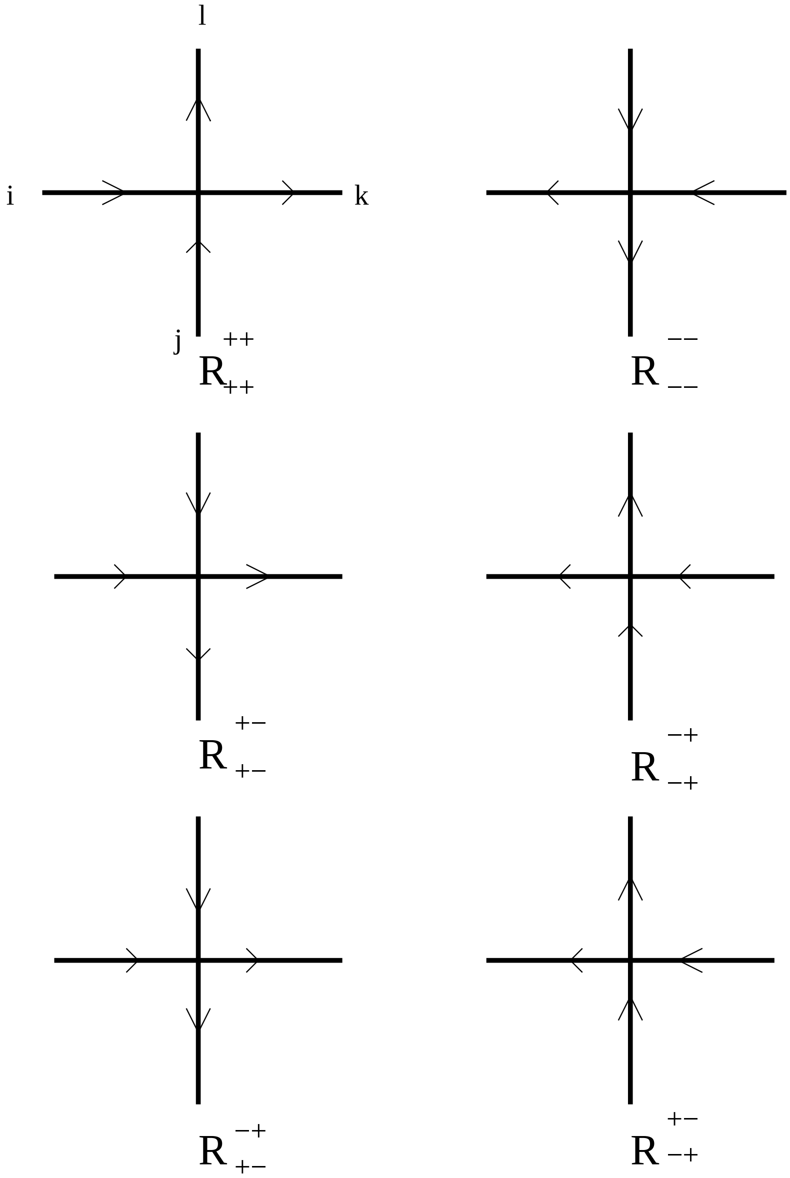
$$R_{++}^{++}(\lambda) = R_{--}^{--}(\lambda) = \sinh \mu(\lambda + i)$$
$$R_{+-}^{-+}(\lambda) = R_{-+}^{+-}(\lambda) = \sinh(\mu\lambda), \qquad R_{+-}^{+-}(\lambda) = R_{-+}^{-+}(\lambda) = \sinh(i\mu)$$

Rewrite the R-matrix in terms of Pauli matrices ($q = e^{i\mu}$):

$$R(\lambda) = \begin{pmatrix} e^{\mu\lambda} q^{\frac{\sigma^z}{2}} - e^{-\mu\lambda} q^{-\frac{\sigma^z}{2}} & (q - q^{-1})\sigma^- \\ (q - q^{-1})\sigma^+ & e^{\mu\lambda} q^{-\frac{\sigma^z}{2}} - e^{-\mu\lambda} q^{\frac{\sigma^z}{2}} \end{pmatrix}$$

The 6-vertex model

'Ice rule': $i + j = k + l$



Relax constraint 8-vertex: $i + j = k + l \; mod(2)$

The Lax operator

The R-matrix in terms of Pauli matrices:

$$R(\lambda) = \begin{pmatrix} e^{\mu\lambda} q^{\frac{\sigma^z}{2}} - e^{-\mu\lambda} q^{-\frac{\sigma^z}{2}} & (q - q^{-1}) e^{\mu\lambda} \sigma^- \\ (q - q^{-1}) e^{-\mu\lambda} \sigma^+ & e^{\mu\lambda} q^{-\frac{\sigma^z}{2}} - e^{-\mu\lambda} q^{\frac{\sigma^z}{2}} \end{pmatrix}$$

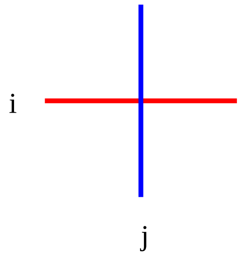
For any representation of $U_q(sl_2)$: $\mathcal{L}$ matrix,

$$\mathcal{L}(\lambda) = \begin{pmatrix} e^{\mu\lambda} \mathrm{A} - e^{-\mu\lambda} \mathrm{D} & (q - q^{-1}) e^{\mu\lambda} \mathrm{B} \\ (q - q^{-1}) e^{-\mu\lambda} \mathrm{C} & e^{\mu\lambda} \mathrm{D} - e^{-\mu\lambda} \mathrm{A} \end{pmatrix}$$

$A, \ B, \ C, \ D$ generate $U_q(sl_2)$:

$$\mathrm{A\ D} = \mathrm{D\ A} = \mathbb{I}, \quad \mathrm{A\ C} = q \mathrm{C\ A}, \quad \mathrm{A\ B} = q^{-1} \mathrm{B\ A},$$
$$\left[\mathrm{C}, \ \mathrm{B} \right] = \frac{\mathrm{A}^2 - \mathrm{D}^2}{q - q^{-1}}.$$

The $\mathcal{L}$ matrix acts on $\mathbb{V} \otimes \mathcal{A}\ (U_q(\widehat{sl_2}))$, $\ q = e^{i\mu}$:



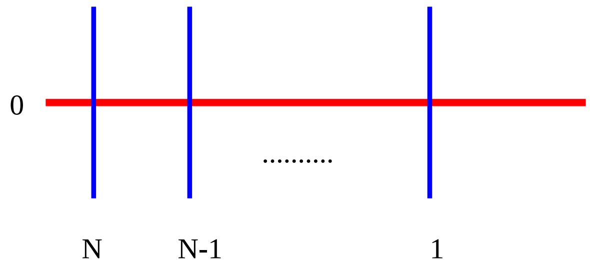
Satisfies the defining relation of $\mathcal{A}$

$$R_{12}(\lambda_1 - \lambda_2)\ \mathcal{L}_{13}(\lambda_1)\ \mathcal{L}_{23}(\lambda_2) = \mathcal{L}_{23}(\lambda_2)\ \mathcal{L}_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

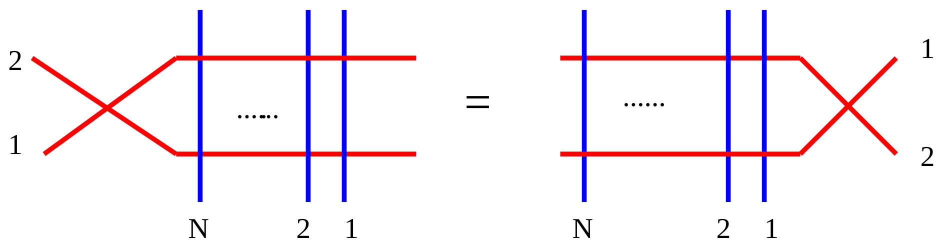
Tensor representations: the periodic spin chain

The monodromy matrix $T \in \text{End}(\mathbb{V}) \otimes \mathcal{A}^{\otimes(N)}$ (QISM: *Faddeev, Takhtajan '81*):

$$T_0(\lambda) = \mathcal{L}_{0N}(\lambda)\ \mathcal{L}_{0\ N-1}(\lambda) \ldots \mathcal{L}_{01}(\lambda)$$



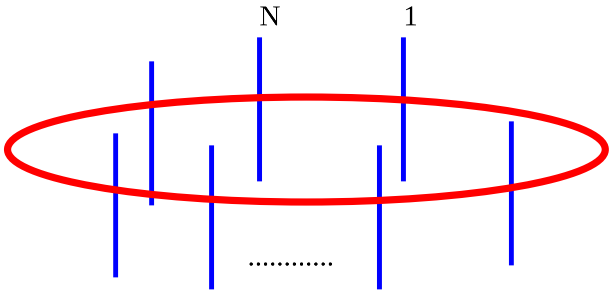
Satisfies the fundamental algebraic relation:



$$R_{12}(\lambda_1 - \lambda_2)\ T_1(\lambda_1)\ T_2(\lambda_2) = T_2(\lambda_2)\ T_1(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

The transfer matrix $t \in \mathcal{A}^{\otimes N}$ (*Faddeev, Takhtajan '81*):

$$t(\lambda) = Tr_0 \, \{T_0(\lambda)\}$$



Provides a family of commuting operators

$$\Big[t(\lambda),\ t(\lambda')\Big] = 0$$

Latter commutation relation ensures Integrability

$$\mathcal{H} \propto \frac{d}{d\lambda}(ln\ t(\lambda))|_{\lambda=0}$$

Bethe ansatz

• Main aim: diagonalization of transfer matrix via algebraic Bethe ansatz.

1. Reference state: highest(lowest) weight ($V^* \; e_1 \; = \; 0!$) pseudo-vacuum state (co-unit in the context Hopf algebras)
2. Use the RTT algebra exchange relations.

• Find the spectrum, analyticity requir. provide BAE.

• BAE important, their solution $\rightarrow$ physically relevant quantities: exact S matrices, thermodynamic properties, correlation functions...

Diagonalization of $t(\lambda)$

The $\mathcal{L}$–matrix rewritten as

$$\mathcal{L}_{0n}(\lambda) = \begin{pmatrix} \alpha_n^+ & \beta_n \\ \gamma_n & \alpha_n^- \end{pmatrix}$$

$$\alpha_n^\pm = \sinh \mu(\lambda \pm i s_n^z), \quad \gamma_n = s_n^+ \sinh i\mu, \quad \beta_n = s_n^- \sinh i\mu$$

Reference state, highest weight:

$$\gamma_n |+\rangle_n = 0, \qquad |\Omega\rangle = \bigotimes_{n=1}^{N} |+\rangle_n \, .$$

and consequently

$$T(\lambda)|\Omega\rangle = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ 0 & \mathcal{D}(\lambda) \end{pmatrix} |\Omega\rangle$$

the diagonal entries of T acting on the pseudovacuum give,

$$\mathcal{A}(\lambda)|\Omega\rangle = \sinh^N \mu(\lambda+is)|\Omega\rangle, \;\; \mathcal{D}(\lambda)|\Omega\rangle = \sinh^N \mu(\lambda-is)|\Omega\rangle$$

Assumption the general Bethe state has the form

$$|\psi\rangle = \mathcal{B}(\lambda_1)\; \mathcal{B}(\lambda_2) \ldots \mathcal{B}(\lambda_M)\; |\Omega\rangle$$

Solve the eigenvalue problem,

$$t(\lambda)|\psi\rangle = \Big(\mathcal{A}(\lambda) + \mathcal{D}(\lambda)\Big)|\psi\rangle = \Lambda(\lambda)|\psi\rangle$$

Commutation relations: $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$, $\mathcal{B}$, from $RTT = TTR$, With the help of $TTR = RTT$ obtain the eigenvalues of $t(\lambda)$

$$\Lambda(\lambda) = \prod_{j=1}^{M} \frac{\sinh \mu(\lambda - \lambda_j - i)}{\sinh \mu(\lambda - \lambda_j)} \sinh^N \mu(\lambda + is) + \prod_{j=1}^{M} \frac{\sinh \mu(\lambda - \lambda_j + i)}{\sinh \mu(\lambda - \lambda_j)} \sinh^N \mu(\lambda - is)$$

The analyticity of $\Lambda \rightarrow \lambda$'s satisfy BAE

$$e_{2s}(\lambda_i) = \prod_{j=1}^{M} e_2(\lambda_i - \lambda_j)$$

where $e_n(\lambda) = \frac{\sinh \mu(\lambda + \frac{in}{2})}{\sinh \mu(\lambda - \frac{in}{2})}$

BAE $\rightarrow$ physical quantities : S-matrix, free energy, specific heat, central charge...

Energy momentum and spin in terms of BA roots

Energy

$$E = -\frac{1}{2\pi} \sum_{j=1}^{M} \frac{\mu \sinh i\mu}{\sinh \mu(\lambda_j + \frac{i}{2}) \sinh \mu(\lambda_j - \frac{i}{2})}$$

Momentum

$$P = -\sum_{j=1}^{M} i \ln \frac{\sinh \mu(\lambda_j + \frac{i}{2})}{\sinh \mu(\lambda_j - \frac{i}{2})}$$

Spin

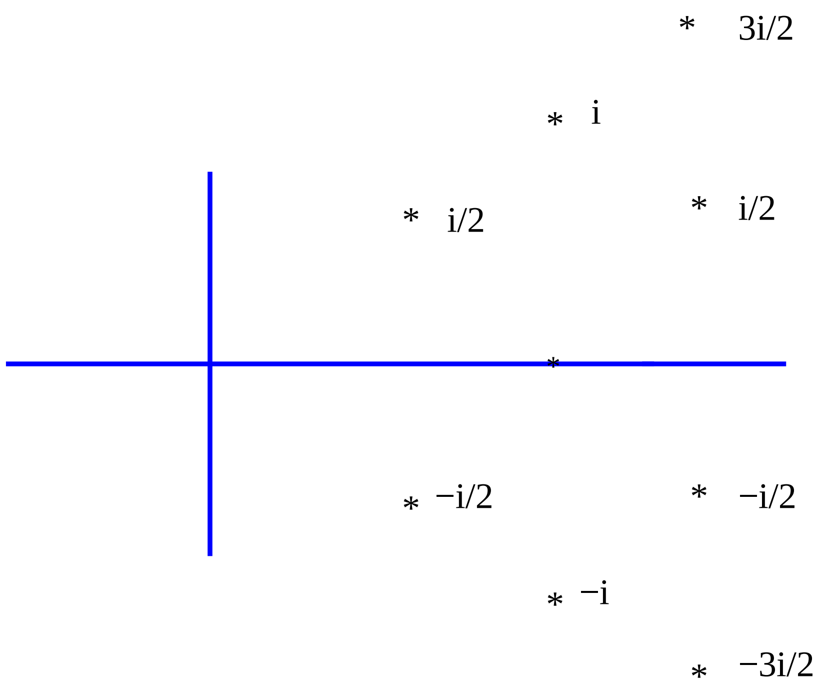
$$S^z = \frac{N}{2} - M$$

Note: $E = \frac{1}{2\pi} \frac{dP}{d\lambda}$

String hypothesis

Solutions of BAE for $N \to \infty$ may be casted as (*Faddeev and Takhtajan '81*):

$$\lambda^{(n,j)} = \lambda_0^{(n)} + \frac{i}{2}(n + 1 - 2j)$$



Solving Bethe ansatz equations

Ground state: all real strings, filled Dirac sea $S^z = 0$.

Low lying excitations: Holes in the filled Dirac sea, particle like excitations

- Spin $S^z = \frac{1}{2}$
- Energy $\epsilon(\lambda) = \frac{1}{2\cosh\frac{\pi\lambda}{2}}$

I hole $\longrightarrow$ 2D rep of $SU(2)$. State with 2 holes, the density (from the BAE):

$$\sigma(\lambda) = 2\pi\epsilon(\lambda) + \frac{1}{N} r(\lambda)$$

$\hat{r}(\omega) = \frac{\sinh(\frac{\pi}{\mu}-2)\frac{\omega}{2}}{2\cosh\frac{\omega}{2}\sinh(\frac{\pi}{\mu}-1)\frac{\omega}{2}}$.

• **Main aim**: derive 2-hole scattering amplitude
Quantization condition (*Korepin '79, Andrei and Destri '84*)

$$(e^{ipN}S - 1)|\lambda_i\rangle = 0$$

recall $\epsilon(\lambda) = \frac{1}{2\pi}\frac{dp(\lambda)}{d\lambda}$, compare the **QC** with the density:

$$S_0(\lambda) = exp\Big[- \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \hat{r}(\omega) e^{-i\omega\lambda}\Big]$$

$$S_0(\lambda) = exp\Big[- \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{\sinh(\frac{\pi}{\mu} - 2)\frac{\omega}{2}}{2\cosh\frac{\omega}{2}\sinh(\frac{\pi}{\mu} - 1)\frac{\omega}{2}} e^{-i\omega\lambda}\Big]$$

sine Gordon S–matrix for $\beta^2 = 8(\pi - \mu)$ (*Zamolodchikov '79*).

More eigenvalues

$$S \propto R$$

We can find all the eigenvalues of the 4×4 S matrix: suitable string configurations

- **2 holes and a 2-string in the middle**

$$S_a(\lambda) = \frac{\sinh \mu(\lambda - i)}{\sinh \mu(\lambda + i)} S_0(\lambda)$$

- **2 holes and a negative parity string in the middle**

$$S_b(\lambda) = \frac{\cosh \mu(\lambda - i)}{\cosh \mu(\lambda + i)} S_0(\lambda)$$

Method applied for higher rank algebras, super–algebras: *Doikou and Nepomechie '97–'99, Doikou '00, Arnaudon, Avan, Crampe, Doikou, Frappat, Ragoucy, '03–'05*

Quantum algebras: deformed co-product

$$\mathcal{L}(\lambda) = e^{\mu\lambda}\mathcal{L}^{+} - e^{-\mu\lambda}\mathcal{L}^{-}$$

$\mathcal{L}^{\pm}_{ab} \in U_q(gl_n)$

As $\lambda \to \pm\infty$ $\mathcal{L}$ and consequently T reduce to upper, lower triangular matrices.

$$T(\lambda \to \pm\infty) \propto T^{\pm},$$

entries of $T^{\pm} \in U_q(gl_n)^{\otimes N}$

e.g. $U_q(sl_2)$

$$\mathcal{L}(\lambda) = e^{\mu\lambda}\begin{pmatrix} q^{s^z} & s^{-} 2\sinh i\mu \\ 0 & q^{-s^z} \end{pmatrix} - e^{-\mu\lambda}\begin{pmatrix} q^{-s^z} & 0 \\ -s^{+} 2\sinh i\mu & q^{s^z} \end{pmatrix}$$

Then asymptotics of T:

$$T^{+} \propto \begin{pmatrix} q^{S^z} & c^{+}S^{-} \\ 0 & q^{-S^z} \end{pmatrix}, \quad T^{-} \propto \begin{pmatrix} q^{-S^z} & 0 \\ c^{-}S^{+} & q^{S^z} \end{pmatrix}$$

$$S^z = \sum_{k=1}^{N} \mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes s_k^z \otimes \mathbb{I} \ldots \otimes \mathbb{I},$$
$$S^{\pm} = \sum_{k=1}^{N} q^{-s_1^z} \otimes \ldots \otimes q^{-s_{k-1}^z} \otimes s_k^{\pm} \otimes q^{s_{k+1}^z} \otimes \ldots \otimes q^{s_N^z}$$

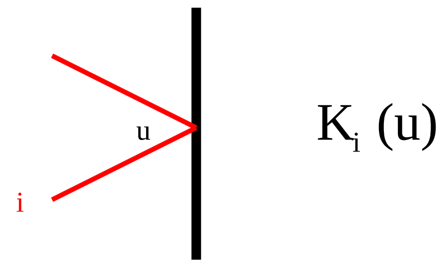
S^z, $S^{\pm}$ tensor product realizations of $U_q(sl_2)$ (*Jimbo '85*)

$$[S^{+},\ S^{-}] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}}, \qquad [S^z,\ S^{\pm}] = \pm S^{\pm}$$

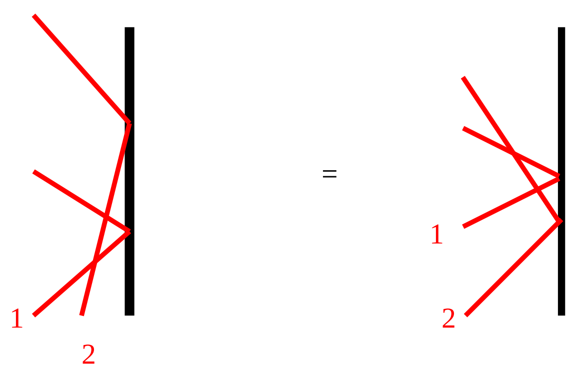
Study of the underlying quantum algerbas (Yangians): $so(n)$, $sp(m)$, $osp(n|m)$, $sl(n|m)$: *Arnaudon, Avan, Crampe, Doikou, Frappat, Ragoucy, '03–'05*. Study of boundary quantum algebras *Doikou,'03-today*

Open boundaries

The K matrix acts on $\mathbb{V}$:



Satisfies the reflection equation (Cherednik '84)



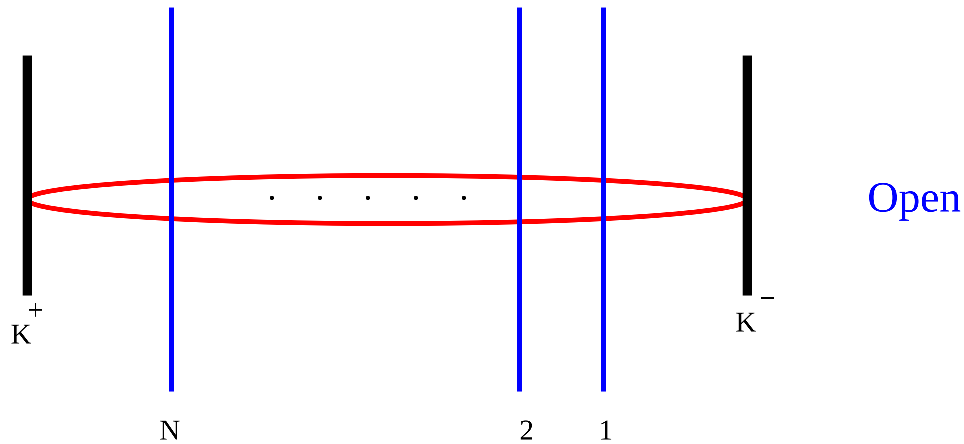
$$\begin{aligned} & R_{12}(\lambda_1 - \lambda_2)\ K_1(\lambda_1)\ R_{21}(\lambda_1 + \lambda_2)\ K_2(\lambda_2) \\ = & K_2(\lambda_2)\ R_{12}(\lambda_1 + \lambda_2)\ K_1(\lambda_1)\ R_{21}(\lambda_1 - \lambda_2) \end{aligned}$$

• Solutions of RE (e.g. via Hecke algebras: Levy and Martin '94, Doikou and Martin '02, Doikou '04) $\longrightarrow$ build open spin chains (Sklyanin '88)

The open spin chain

Integrable boundary conditions (Sklyanin '88)

$$t(\lambda) = tr_0\ K_0^{(l)}(\lambda)\ \underbrace{T_0(\lambda)\ K_0^{(r)}(\lambda)\ T_0^{-1}(-\lambda)}_{\mathcal{T}_0(\lambda)}$$



$$\Big[t(\lambda),\ t(\lambda')\Big] = 0$$

Integrability ensured.
Boundary S matrices (Doikou, Mezincescu and Nepomechie '97)
Boundary symmetries (Doikou '04).