On Liouville integrable defects

Anastasia Doikou

University of Patras

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- Work in collaboration with J. Avan.

General frame

- Integrable defects (quantum level) impose severe constraints on relevant algebraic and physical quantities (e.g. scattering amplitudes) (*Delfino, Mussardo, Simonetti, Konic, LeClair, ...*)
- In discrete integrable systems there is a systematic description of local defects based on QISM
- In integrable field theories a defect is introduced as discontinuity plus gluing conditions (*Bowcock, Corrigan, Zambon,...*), integrability issue not systematically addressed; other attempts (*Caudrelier, Kundu, Habibulin,...*)
- We developed a systematic *algebraic* means to investigate integrable filed theories with point like defects. Integrability is ensured by construction

Outline

1. The general frame
   - The L matrix
   - The classical quadratic algebra
2. Local integrals of motion, and relevant Lax pairs for NLS and sine-Gordon models
3. Discrete theories and consistent continuum limits
4. Discussion and future perspectives

Classical Integrability

The Lax pair $\mathbb{U}$, $\mathbb{V}$; the linear auxiliary problem (e.g. *Faddeev-Takhtajan*):

$$\frac{\partial \Psi(x,t)}{\partial x} = \mathbb{U}(x,t)\ \Psi(x,t)$$
$$\frac{\partial \Psi(x,t)}{\partial t} = \mathbb{V}(x,t)\ \Psi(x,t)$$

Compatibility condition leads to

Zero curvature condition

$$\dot{\mathbb{U}}(x,t) - \mathbb{V}'(x,t) + \Big[\mathbb{U}(x,t), \mathbb{V}(x,t)\Big] = 0$$

Gives rise to the equations of motion of the system.

The monodromy matrix

The continuum monodromy matrix

$$T(x, y, \lambda) = P \exp\left\{ \int_x^y dx \ \mathbb{U}(x) \right\}$$

Solution of the differential equation

$$\frac{\partial T(x, y)}{\partial x} = \mathbb{U}(x, t) \ T(x, y)$$

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$$\frac{\partial T(x, y)}{\partial x} = \mathbb{U}(x, t) \ T(x, y)$$

$\mathbb{U}$ obeys linear classical algebra, T satisfies the:

Classical algebra

$$\left\{ T_a(\lambda), \ T_b(\mu) \right\} = \left[r_{ab}(\lambda - \mu), \ T_a(\lambda) \ T_b(\mu) \right]$$

The classical r-matrix satisfies the CYBE (*Sklyanin, Semenov-Tian-Shansky*)

$$\boxed{[r_{12}, \ r_{13}] + [r_{12}, \ r_{23}] + [r_{13}, \ r_{23}] = 0.}$$

Classical integrability

The monodromy matrix T satisfies the classical algebra, thus

The transfer matrix

$$t(\lambda) = Tr\ T(\lambda)$$

provides the charges in involution;

$$\Big\{t(\lambda),\ t(\mu)\Big\} = 0$$

integrability ensured by construction. In $t(\lambda)\ \rightarrow$ *local* integrals of motion

The defect frame

The key object, modified monodromy:

Defect monodromy matrix

$$T(L,-L,\lambda) = T^{+}(L,x_0,\lambda)\ \tilde{L}(x_0,\lambda)\ T^{-}(x_0,-L,\lambda)$$

where we define

$$T^{\pm} = P\exp\Big\{\int dx\ \mathbb{U}^{\pm}(x)\Big\}$$

The defect $\tilde{L}$ matrix obeys

$$\Big\{\tilde{L}_a(\lambda_1),\ \tilde{L}_b(\lambda_2)\Big\} = \Big[r_{ab}(\lambda_1-\lambda_2),\ L_a(\lambda_1)\ L_b(\lambda_2)\Big]$$

$T^{\pm}$ satisfy the classical algebra, thus T obeys the same algebra, integrability also ensured

The defect frame

Auxiliary linear problem for $\mathbb{U}^{\pm}$, $\mathbb{V}^{\pm}$ for the defect theory:

$$\frac{\partial \Psi(x,t)}{\partial x} = \mathbb{U}^{\pm}\ \Psi(x,t)$$
$$\frac{\partial \Psi(x,t)}{\partial t} = \mathbb{V}^{\pm}\ \Psi(x,t)$$

The corresponding

Zero curvature condition

$$\dot{\mathbb{U}}^{\pm}(x,t) - \mathbb{V}^{\pm\prime}(x,t) + \Big[\mathbb{U}^{\pm}(x,t), \mathbb{V}^{\pm}(x,t)\Big] = 0 \qquad x \neq x_0$$

The defect frame

Auxiliary linear problem for $\mathbb{U}^{\pm}$, $\mathbb{V}^{\pm}$ for the defect theory:

$$\frac{\partial \Psi(x,t)}{\partial x} = \mathbb{U}^{\pm}\ \Psi(x,t)$$
$$\frac{\partial \Psi(x,t)}{\partial t} = \mathbb{V}^{\pm}\ \Psi(x,t)$$

The corresponding

Zero curvature condition

$$\dot{\mathbb{U}}^{\pm}(x,t) - \mathbb{V}^{\pm\,\prime}(x,t) + \Big[\mathbb{U}^{\pm}(x,t), \mathbb{V}^{\pm}(x,t)\Big] = 0 \qquad x \neq x_0$$

On the defect point

Defect zero curvature condition

$$\frac{d\tilde{L}(x_0)}{dt} = \tilde{\mathbb{V}}^{+}(x_0)\tilde{L}(x_0) - \tilde{L}(x_0)\tilde{\mathbb{V}}^{-}(x_0)$$

The NLS model with defect

The $\mathbb{U}^{\pm}$-operator for the NLS model:

$$\mathbb{U}^{\pm} = \frac{\lambda}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & \bar{\psi}^{\pm} \\ \psi^{\pm} & 0 \end{pmatrix}.$$

From the classical algebra for $\mathbb{U}$:

Poisson structure

$$\Big\{\psi^{\pm}(x),\ \bar{\psi}^{\pm}(y)\Big\} = \delta(x-y), \qquad \Big\{\psi^{\mp}(x),\ \bar{\psi}^{\pm}(y)\Big\} = 0.$$

The classical r-matrix is the Yangian: $r(\lambda) = \frac{1}{\lambda}\mathcal{P}$ (*Yang*)
$\mathcal{P}(a \otimes b) = b \otimes a$.

The NLS model with defect

The generic defect $\tilde{L}$ operator

$$\tilde{L}(x_0) = \lambda \mathbb{I} + \begin{pmatrix} \alpha(x_0) & \beta(x_0) \\ \gamma(x_0) & \delta(x_0) \end{pmatrix}.$$

From the quadratic classical algebra for $\tilde{L}$ ($\mathfrak{sl}_2$ algebra):

$$\Big\{\alpha(x_0),\ \beta(x_0)\Big\} = \beta(x_0)$$
$$\Big\{\alpha(x_0),\ \gamma(x_0)\Big\} = -\gamma(x_0)$$
$$\Big\{\beta(x_0),\ \gamma(x_0)\Big\} = 2\alpha(x_0)$$

Establish the Poisson structure!

- Relevant studies: (*Corrigan-Zambon*)

The NLS model: local IM

First recall that:

$$\frac{\partial T^{\pm}(x,y,t)}{\partial x} = \mathbb{U}^{\pm}(x,t)\ T^{\pm}(x,y,t)$$

Based on the latter consider the decomposition ansatz:

$$T^{\pm}(x,y;\lambda) = (1+W^{\pm}(x))e^{Z^{\pm}(x,y)}(1+W^{\pm}(y))^{-1}$$

$W^{\pm}$ anti-diagonal, $Z^{\pm}$ diagonal. Also,

$$W^{\pm} = \sum_{n=0}^{\infty}\frac{W^{\pm(n)}}{\lambda^n}, \qquad Z^{\pm} = \sum_{n=-1}^{\infty}\frac{Z^{\pm(n)}}{\lambda^n}$$

Substituting the ansatz to the differential equation above identify $W^{\pm(n)}$, $Z^{\pm(n)}$ matrices.

The NLS model: local IM

Substitution leads to *Riccati-type*:

Differential equations

$$\frac{dW^{\pm}}{dx} + W^{\pm}\mathbb{U}_d - \mathbb{U}_d W^{\pm} + W^{\pm}\mathbb{U}_a^{\pm} W^{\pm} - \mathbb{U}_a^{\pm} = 0$$

$$\frac{dZ^{\pm}}{dx} = \mathbb{U}_d + \mathbb{U}_a^{\pm} W^{\pm}$$

Solving the latter one identifies the $W^{\pm(n)}$, $Z^{\pm(n)}$, hence the charges in involution.

- Similar differential equations arise within the inverse scattering frame.

The NLS model: local IM

The generating function

$$\mathcal{G}(\lambda) = \ln tr(T^{+}(\lambda)\ \tilde{L}(\lambda, x_0)\ T^{-}(\lambda))$$

which turns to, via the decomposition:

Generating function

$$\mathcal{G}(\lambda) = Z_{11}^{+}(\lambda) + Z_{11}^{-}(\lambda) + \ln[(1 + W^{+}(x_0))^{-1}\tilde{L}(x_0)(1 + W^{-}(x_0))]_{11}$$

Also,

$$\mathcal{G}(\lambda) = \sum_{n=0}^{\infty} \frac{\mathcal{H}^{(n)}}{\lambda^n}$$

The NLS model: local IM

The first three integrals of motion:

- **The number of particles**

$$\mathcal{H}^{(1)} = \int_{-L}^{x_0^-} dx\ \psi^-(x)\bar{\psi}^-(x) + \int_{x_0^+}^{L} dx\ \psi^+(x)\bar{\psi}^+(x) + \alpha(x_0)$$

- **The momentum**

$$\begin{aligned}\mathcal{H}^{(2)} &= -\int_{-L}^{x_0^-} dx\ \bar{\psi}^-(x)\psi^{-'}(x) - \int_{x_0^+}^{L} dx\ \bar{\psi}^+(x)\psi^{+'}(x) \\ &- \quad \bar{\psi}^+\psi^+ + \bar{\psi}^+\psi^- + \gamma\bar{\psi}^+ + \beta\psi^- - \frac{\alpha^2}{2}\end{aligned}$$

- **The Hamiltonian**

$$
\begin{aligned}
\mathcal{H}^{(3)} &= \int_{x_0^+}^{L} dx\, \left(\bar{\psi}^+ \psi^{+\prime\prime} + |\psi^+|^4\right) + \int_{-L}^{x_0^-} dx\, \left(\bar{\psi}^- \psi^{-\prime\prime} + |\psi^-|^4\right) \\
&+ (\bar{\psi}^+\psi^+)' + \gamma\bar{\psi}^{+\prime} - \beta\psi^{-\prime} + \bar{\psi}^{+\prime}\psi^- + \frac{\alpha^3}{3} \\
&- \bar{\psi}^+\psi^{-\prime} - \alpha\Big(\gamma\bar{\psi}^+ + \beta\psi^- + 2\bar{\psi}^+\psi^-\Big)
\end{aligned}
$$

- **The Hamiltonian**

$$\begin{aligned}\mathcal{H}^{(3)} &= \int_{x_0^+}^{L} dx\ \Big(\bar{\psi}^+ \psi^{+''} + |\psi^+|^4\Big) + \int_{-L}^{x_0^-} dx\ \Big(\bar{\psi}^- \psi^{-''} + |\psi^-|^4\Big) \\ &+ (\bar{\psi}^+\psi^+)' + \gamma \bar{\psi}^{+'} - \beta \psi^{-'} + \bar{\psi}^{+'} \psi^- + \frac{\alpha^3}{3} \\ &- \bar{\psi}^+ \psi^{-'} - \alpha\Big(\gamma \bar{\psi}^+ + \beta \psi^- + 2\bar{\psi}^+ \psi^-\Big)\end{aligned}$$

By construction (formally), and also explicitly checked:

Involution

$$\Big\{\mathcal{H}_1,\ \mathcal{H}_2\Big\} = \Big\{\mathcal{H}_1,\ \mathcal{H}_3\Big\} = \Big\{\mathcal{H}_2,\ \mathcal{H}_3\Big\} = 0$$

No sewing constraints arise or used so far, *off-shell* integrability.

The NLS model: Lax pair

Next step, derive time component of the Lax pair $\mathbb{V}$, and sewing conditions. Explicit expressions (*Faddeev-Takhtajan, Avan-Doikou*):

$$\mathbb{V}^{+}(x,\lambda,\mu) = t^{-1} tr_a\Big(T_a^{+}(L,x) r_{ab}(\lambda-\mu) T_a^{+}(x,x_0)\tilde{L}_a(x_0) T_a^{-}(x_0,-L)\Big)$$

$$\mathbb{V}^{-}(x,\lambda,\mu) = t^{-1} tr_a\Big(T_a^{+}(L,x_0)\tilde{L}_a(x_0) T_a^{-}(x_0,x) r_{ab}(\lambda-\mu) T_a^{-}(x,-L)\Big)$$

$$\tilde{\mathbb{V}}^{+}(x_0,\lambda,\mu) = t^{-1} tr_a\Big(T_a^{+}(L,x_0) r_{ab}(\lambda-\mu)\tilde{L}_a(x_0) T_a^{-}(x_0,-L)\Big)$$

$$\tilde{\mathbb{V}}^{-}(x_0,\lambda,\mu) = t^{-1} tr_a\Big(T_a^{+}(L,x_0)\tilde{L}_a(x_0) r_{ab}(\lambda-\mu) T_a^{-}(x_0,-L)\Big).$$

The NLS model: Lax pair

For the left and right bulk theories, and the defect point:

$$\mathbb{V}^{\pm(1)}(\mu,\ x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbb{V}^{-(2)}(\mu,\ x) = \begin{pmatrix} \mu & \bar{\psi}^{-}(x) \\ \psi^{-}(x) & 0 \end{pmatrix}$$

$$\mathbb{V}^{+(2)}(\mu,\ x) = \begin{pmatrix} \mu & \bar{\psi}^{+}(x) \\ \psi^{+}(x) & 0 \end{pmatrix}$$

$$\tilde{\mathbb{V}}^{-(2)}(\mu,\ x_0) = \begin{pmatrix} \mu & \bar{\psi}^{+}(x_0) + \beta(x_0) \\ \psi^{-}(x_0) & 0 \end{pmatrix}$$

$$\tilde{\mathbb{V}}^{+(2)}(\mu,\ x_0) = \begin{pmatrix} \mu & \bar{\psi}^{+}(x_0) \\ \gamma(x_0) + \psi^{-}(x_0) & 0 \end{pmatrix}$$

The NLS model: Lax pair

$$\mathbb{V}^{-(3)}(\mu,\ x) = \begin{pmatrix} \mu^2 - \bar{\psi}^-(x)\psi^-(x) & \mu\bar{\psi}^-(x) + \bar{\psi}^{-'}(x) \\ \mu\psi^-(x) - \psi^{-'}(x) & \bar{\psi}^-(x)\psi^-(x) \end{pmatrix}$$

$$\mathbb{V}^{+(3)}(\mu,\ x) = \begin{pmatrix} \mu^2 - \bar{\psi}^+(x)\psi^+(x) & \mu\bar{\psi}^+(x) + \bar{\psi}^{+'}(x) \\ \mu\psi^+(x) - \psi^{+'}(x) & \bar{\psi}^+(x)\psi^+(x) \end{pmatrix}$$

$$\tilde{\mathbb{V}}^{-(3)}(x_0) = \begin{pmatrix} \mu^2 - \Big(\bar{\psi}^+(x_0) + \beta(x_0)\Big)\psi^-(x_0) & \mu\Big(\bar{\psi}^+(x_0) + \beta(x_0)\Big) + \mathfrak{f}(x_0) \\ \mu\psi^-(x_0) - \psi^{-'}(x_0) & \Big(\bar{\psi}^+(x_0) + \beta(x_0)\Big)\psi^-(x_0) \end{pmatrix}$$

$$\tilde{\mathbb{V}}^{+(3)}(x_0) = \begin{pmatrix} \mu^2 - \bar{\psi}^+(x_0)(\psi^-(x_0) + \gamma(x_0)) & \mu\bar{\psi}^+(x_0) + \bar{\psi}^{+'}(x_0) \\ \mu\Big(\psi^-(x) + \gamma(x_0)\Big) + \mathfrak{g}(x_0) & \bar{\psi}^+(x_0)\Big(\psi^-(x_0) + \gamma(x_0)\Big) \end{pmatrix}$$

where we define

$$\mathfrak{f}(x_0) = \bar{\psi}^{+'}(x_0) - \alpha(x_0)\Big(\beta(x_0) + 2\bar{\psi}^+(x_0)\Big)$$

$$\mathfrak{g}(x_0) = -\psi^{-'}(x_0) - \alpha(x_0)\Big(\gamma(x_0) + 2\psi^-(x_0)\Big)$$

Derive the equations of motion via the Hamiltonian:

Equations of motion

$$\dot{\psi}^{\pm}(x,t) = \{\mathcal{H}^{(j)},\ \psi^{\pm}(x,t)\}, \qquad \dot{\bar{\psi}}^{\pm}(x,t) = \{\mathcal{H}^{(j)},\ \bar{\psi}^{\pm}(x,t)\}$$
$$\dot{\mathrm{e}}(x_0,t) = \{\mathcal{H}^{(j)},\ \mathrm{e}(x_0,t)\}, \qquad \mathrm{e} \in \{\alpha,\ \beta,\ \gamma\}, \qquad x \neq x_0$$

The NLS model: Equations of motion

Derive the equations of motion via the Hamiltonian:

Equations of motion

$$\dot{\psi}^{\pm}(x,t) = \{\mathcal{H}^{(j)},\ \psi^{\pm}(x,t)\}, \qquad \dot{\bar{\psi}}^{\pm}(x,t) = \{\mathcal{H}^{(j)},\ \bar{\psi}^{\pm}(x,t)\}$$

$$\dot{\mathrm{e}}(x_0,t) = \{\mathcal{H}^{(j)},\ \mathrm{e}(x_0,t)\}, \qquad \mathrm{e} \in \{\alpha,\ \beta,\ \gamma\}, \qquad x \neq x_0$$

The latter lead to the familiar E.M. for $\mathcal{H}^{(3)}$

$$\dot{\psi}^{\pm}(x,t) = \frac{\partial^2 \psi^{\pm}(x,t)}{\partial x^2} - 2|\psi^{\pm}(x,t)|^2 \psi^{\pm}(x,t)$$

$$\dot{\bar{\psi}}^{\pm}(x,t) = \frac{\partial^2 \bar{\psi}^{\pm}(x,t)}{\partial x^2} - 2|\psi^{\pm}(x,t)|^2 \bar{\psi}^{\pm}(x,t)$$

Plus E.M. on the defect point. Consistency check via the zero curvature condition.

The NLS model: sewing conditions

Due to continuity requirements at the points x_0^+, x_0^-:

Continuity

$$\mathbb{V}^{+(k)}(x_0^+) \to \tilde{\mathbb{V}}^{+(k)}(x_0), \qquad \mathbb{V}^{-(k)}(x_0^-) \to \tilde{\mathbb{V}}^{-(k)}(x_0), \qquad x_0^{\pm} \to x_0$$

The NLS model: sewing conditions

Due to continuity requirements at the points $x_0^+, \ x_0^-$:

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$$\mathbb{V}^{+(k)}(x_0^+) \to \tilde{\mathbb{V}}^{+(k)}(x_0), \qquad \mathbb{V}^{-(k)}(x_0^-) \to \tilde{\mathbb{V}}^{-(k)}(x_0), \qquad x_0^\pm \to x_0$$

The following sewing conditions $C_\pm^{(k)}$ arise

$$\begin{aligned}
C_-^{(1)}: &\qquad \bar{\psi}^-(x_0) - \bar{\psi}^+(x_0) - \beta(x_0) = 0, \\
C_+^{(1)}: &\qquad \psi^+(x_0) - \psi^-(x_0) - \gamma(x_0) = 0 \\
C_-^{(2)}: &\qquad \bar{\psi}^{-\prime}(x_0) - \bar{\psi}^{+\prime}(x_0) + \alpha(x_0)\beta(x_0) + 2\alpha(x_0)\bar{\psi}^+(x_0) = 0 \\
C_+^{(2)}: &\qquad \psi^{-\prime}(x_0) - \psi^{+\prime}(x_0) + \alpha(x_0)\gamma(x_0) + 2\alpha(x_0)\psi^-(x_0) = 0 \\
&\quad \ldots
\end{aligned}$$

Jump across the defect point!

The NLS model: sewing conditions

Main proposition:

Compatibility

$$\Big\{\mathcal{H}^{(k)},\ \mathcal{C}_{\pm}^{(m,l)}\Big\} = \sum_{i=0}^{k-1}\Big[\mathcal{C}_{\pm}^{(k,i)},\ \mathbb{V}^{\pm(m+i,l)}(x_0^{\pm})\Big] + \sum_{i=0}^{k-1}\Big[\tilde{\mathbb{V}}^{\pm(k,i)}(x_0),\ \mathcal{C}_{\pm}^{(m+i,l)}\Big]$$

- $\mathcal{C}_{\pm}^{(p,l)}$ matrices with entries the constraints. Proof based on the form of $\mathbb{V}$, and the underlying algebra.
- Sub-manifold of sewing conditions (dynamical constraints) invariant under the Hamiltonian action!

The sine-Gordon model with defect

The $\mathbb{U}$-operator for the sine-Gordon model:

$$\mathbb{U}(x,t,u) = \frac{\beta}{4i}\pi(x,t)\sigma^z + \frac{mu}{4i}e^{\frac{i\beta}{4}\phi\sigma^z}\sigma^y e^{-\frac{i\beta}{4}\phi\sigma^z} - \frac{mu^{-1}}{4i}e^{-\frac{i\beta}{4}\phi\sigma^z}\sigma^y e^{\frac{i\beta}{4}\phi\sigma^z}$$

$u \equiv e^{\lambda}$, $\sigma^{x,y,z}$ Pauli matrices. The r-matrix (*Faddeev-Takhtajan, Sklyanin*):

$$r(\lambda) = \frac{\beta^2}{8\sinh\lambda}\begin{pmatrix} \frac{\sigma^z+1}{2}\cosh\lambda & \sigma^- \\ \sigma^+ & \frac{-\sigma^z+1}{2}\cosh\lambda \end{pmatrix}.$$

$\mathbb{U}$ satisfies the linear Poisson algebra leads:

$$\Big\{\phi(x),\ \pi(y)\Big\} = \delta(x-y)$$

The sine-Gordon model with defect

The relevant defect matrix (type II)

$$\tilde{L}(\lambda) = \begin{pmatrix} e^{\lambda} V - e^{-\lambda} V^{-1} & \bar{a} \\ a & e^{\lambda} V^{-1} - e^{-\lambda} V \end{pmatrix}.$$

$\tilde{L}$ satisfies the classical algebra, hence:

$$\left\{V,\ \bar{a}\right\} = \frac{\beta^2}{8} V\ \bar{a},$$

$$\left\{V,\ a\right\} = -\frac{\beta^2}{8} Va,$$

$$\left\{\bar{a},\ a\right\} = \frac{\beta^2}{4}(V^2 - V^{-2})$$

- Relevant studies: (*Bowcock-Corrigan-Zambon, Caudrelier, Habibulin-Kundu, Aguirre etal.*)

The sine-Gordon: local IM

Recall the generating function of the local IM

$$\mathcal{G}(\lambda) = \ln tr(T^{+}\ \tilde{L}\ T^{-})$$

Generating function

$$\mathcal{G}(\lambda) = Z^{+}_{11} + Z^{-}_{11} + \ln\Big[(1+W^{+})^{-1}(\Omega^{+}(x_0))^{-1}\tilde{L}(x_0)\Omega^{-}(x_0)(1+W^{-})\Big]_{11}$$

$\Omega^{\pm} = e^{\frac{i\beta}{4}\phi^{\pm}\sigma^{z}}$.

Expanding the latter expression in powers of u^{-1} we obtain the following:

$$\mathcal{G}(\lambda) = \sum_{m=0}^{\infty} \frac{\mathcal{I}^{(m)}}{u^{m}}$$

The sine-Gordon: local IM

The first charge (the u^{-1}-expansion) leads to $\mathcal{I}^{(1)}$, the u-expansion leads to $\mathcal{I}^{(-1)}$:

$$\mathcal{I}^{(-1)}(\phi,\ \pi,\ V,\ a, \bar{a}) = \mathcal{I}^{(1)}(-\phi,\ \pi,\ V^{-1},\ a, \bar{a})$$

Define the

Hamiltonian

$$\begin{aligned}
\mathcal{H} &= \frac{2im}{\beta^2}(\mathcal{I}^{(1)} - \mathcal{I}^{(-1)}) \\
&= \int_{-L}^{x_0^-} dx \left(\frac{1}{2}(\pi^{-2}(x) + \phi^{-\prime 2}(x)) - \frac{m^2}{\beta^2}\cos(\beta\phi^-(x))\right) \\
&+ \int_{x_0^+}^{L} dx \left(\frac{1}{2}(\pi^{+2}(x) + \phi^{+\prime 2}(x)) - \frac{m^2}{\beta^2}\cos(\beta\phi^+(x))\right) \\
&+ \frac{4m}{\beta^2\mathcal{D}}\cos\frac{\beta}{4}(\phi^+ + \phi^-)\left(\bar{a} - a\right) + \frac{2i}{\beta\mathcal{D}}\left(\phi^{+\prime} + \phi^{-\prime}\right)\mathcal{A}
\end{aligned}$$

The sine-Gordon: local IM

Also derive the:

Momentum

$$\mathcal{P} = \frac{2im}{\beta^2}\Big(\mathcal{I}^{(1)} + \mathcal{I}^{(-1)}\Big)$$
$$= \int_{-L}^{x_0^-} dx\ \phi^{-'}(x)\pi^-(x) + \int_{x_0^+}^{L} dx\ \phi^{+'}(x)\pi^+(x)$$
$$- \frac{4mi}{\beta^2\mathcal{D}} \sin\frac{\beta}{4}(\phi^+ + \phi^-)\ \Big(\bar{a} + a\Big) + \frac{2i}{\beta\mathcal{D}}\Big(\pi^+ + \pi^-\Big)\mathcal{A}$$

$$\mathcal{D} = Ve^{-\frac{i\beta}{4}(\phi^+ - \phi -)} + V^{-1}e^{\frac{i\beta}{4}(\phi^+ - \phi -)}$$
$$\mathcal{A} = Ve^{-\frac{i\beta}{4}(\phi^+ - \phi -)} - V^{-1}e^{\frac{i\beta}{4}(\phi^+ - \phi -)}$$

- Commutativity among the IM will be discussed later.

The sine-Gordon: the Lax pair

From the explicit expression for the $\mathbb{V}$ operators we find or the left and right bulk theories:

$$\mathbb{V}^{\pm}_{\mathcal{H}} = \frac{\beta}{4i}\phi^{\pm'}\sigma^{z} + \frac{vm}{4i}\Omega^{\pm}\sigma^{y}(\Omega^{\pm})^{-1} + \frac{v^{-1}m}{4i}(\Omega^{\pm})^{-1}\sigma^{y}\Omega^{\pm}$$

$$\mathbb{V}^{\pm}_{\mathcal{P}} = \frac{\beta}{4i}\pi^{\pm}\sigma^{z} + \frac{vm}{4i}\Omega^{\pm}\sigma^{y}(\Omega^{\pm})^{-1} - \frac{v^{-1}m}{4i}(\Omega^{\pm})^{-1}\sigma^{y}\Omega^{\pm}$$

Computation of the $\mathbb{V}$ operators on the defect point leads to

$$\tilde{\mathbb{V}}^{\pm}_{\mathcal{H}} = \mathbb{V}^{\pm}_{\mathcal{H}} + \delta_{\mathcal{H}}$$
$$\tilde{\mathbb{V}}^{\pm}_{\mathcal{P}} = \mathbb{V}^{\pm}_{\mathcal{P}} + \delta_{\mathcal{P}}$$

The sine-Gordon: sewing conditions

Continuity requirements around the defect point

$$\delta_{\mathcal{H}} \to 0, \qquad \delta_{\mathcal{P}} \to 0$$

lead to:

Sewing conditions

$$S_1: \qquad V = e^{\frac{i\beta}{4}(\phi^+ - \phi^-)}$$

$$S_2: \qquad \pi^+(x_0) - \pi^-(x_0) = \frac{im}{\beta} \cos \frac{\beta}{4}(\phi^+(x_0) + \phi^-(x_0)) \left(a + \bar{a}\right)$$

$$S_2': \qquad {\phi^+}'(x_0) - {\phi^-}'(x_0) = \frac{m}{\beta} \sin \frac{\beta}{4}(\phi^+(x_0) + \phi^-(x_0)) \left(\bar{a} - a\right)$$

- *Jump* across the defect point!
- E.M. from Hamiltonian and zero curvature condition coincide.

The sine-Gordon: commutativity

Commutativity among the IM, explicitly checked, formally guaranteed

Commutativity

$$\Big\{\mathcal{H},\ \mathcal{P}\Big\} = 0$$

- The latter is proven using the sewing conditions, i.e. Dirac (not Poisson) commutativity! *On-shell* integrability.
- In NLS *off-shell* integrability

$$\Big\{\mathcal{I}_1,\ \mathcal{I}_2\Big\} = 0$$

no use of constraints. Issue related to suitable continuum limits!

The continuum limit

Integrable continuum limit (see e.g. *Avan-Doikou-Sfetsos*): The discrete monodromy matrix:

$$T_0(\lambda) = L_{0N}(\lambda) \dots L_{02}(\lambda)\ L_{01}(\lambda)$$

L and T satisfy the classical quadratic algebra

$$\Big\{L_a(\lambda,\ L_b(\lambda')\Big\} = \Big[r_{ab}(\lambda-\lambda'),\ L_a(\lambda)L_b(\lambda')\Big]$$

Hence, integrability is guaranteed!

The continuum limit

Consider the identifications:

$$L_n \to 1 + \delta \mathbb{U}(x), \qquad \mathbb{A}_n \to \mathbb{V}(x), \qquad \mathbb{A}_{n+1} \to \mathbb{V}(x+\delta)$$

The *discrete* zero curvature condition:

$$\dot{L}_j = \mathbb{A}_{j+1}\ L_j - L_j\ \mathbb{A}_j$$

takes the familiar continuum form:

Continuum zero curvature

$$\dot{\mathbb{U}} - \mathbb{V}' + \Big[\mathbb{U},\ \mathbb{V}\Big] = 0$$

We have kept terms proportional to δ in the discrete zero curvature condition.

The continuum limit

Recall

$$L_{ai} = 1 + \delta \mathbb{U}_{ai} + \mathcal{O}(\delta^2) \ ,$$

Then the monodromy matrix is expanded as:

$$T_a = 1 + \delta \sum_i \mathbb{U}_{ai} + \delta^2 \sum_{i<j} \mathbb{U}_{a_i} \mathbb{U}_{aj} + \dots \ .$$

Use also

$$\delta \sum_{j=1} f_j \ \rightarrow \ \int_{-L}^{L} dx \ f(x)$$

which leads to the familiar continuum expression

The continuum monodromy

$$T(\lambda) = P \exp\left(\int_{-L}^{L} dx \ \mathbb{U}(x) \right)$$

The continuum limit

Then the discrete monodromy matrix in the *presence of defect*:

$$T_a(\lambda) = L_{aN}(\lambda) \dots \tilde{L}_{an}(\lambda) \dots L_{a1}(\lambda)$$

according to previous analysis T will be formally expressed at the continuum limit:

The defect monodromy

$$T(\lambda) = P \exp\Big(\int_{-L}^{x_0^-} dx\ \mathbb{U}^-(x)\Big)\ \tilde{L}(\lambda, x_0)\ P \exp\Big(\int_{x_0^+}^{L} dx\ \mathbb{U}^+(x)\Big)$$

The continuum limit

The zero curvature condition for the left-right bulk theories:

$$\dot{\mathbb{U}}^{\pm} - \mathbb{V}^{\pm\prime} + \left[\mathbb{U}^{\pm},\ \mathbb{V}^{\pm}\right] = 0$$

The zero curvature condition on the *defect point*, discrete:

$$\dot{\tilde{L}}_n(\lambda) = \mathbb{A}_{n+1}\ \tilde{L}_n(\lambda) - \tilde{L}_n(\lambda)\ \mathbb{A}_n(\lambda)$$

Recalling the latter identifications obtain:

Continuum limit

$$\dot{\tilde{L}}(x_0, \lambda) = \mathbb{V}^+(x_0, \lambda)\ \tilde{L}(x_0, \lambda) - \tilde{L}(x_0, \lambda)\ \mathbb{V}^-(x_0, \lambda)$$

- In NLS: (*Doikou, Avan-Doikou*)

The discrete NLS

The DNLS L-matrix

$$L(\lambda) = \begin{pmatrix} 1+\delta\lambda-\delta^2 xX & \Delta x \\ -\delta X & 1 \end{pmatrix}$$

$$\tilde{L}(\lambda) = \delta\lambda + \delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Introduce:

$$\begin{array}{llll} x_j \to \bar{\psi}^-(x), & X_j \to -\psi^-(x), & 1 \le j \le n-1, & x \in (-L,\ x_0) \\ x_j \to \bar{\psi}^+(x), & X_j \to -\psi^+(x), & n+1 \le j \le N, & x \in (x_0,\ L) \end{array}$$

Obtain the continuum NLS $\mathbb{U}$-matrix.

- Discrete NLS: (*Doikou*)

The discrete NLS

The first IM for DNLS:

$$H^{(1)} = -\sum_{j\neq n} x_j X_j + \alpha_n$$

$$H^{(2)} = -\sum_{j\neq n,n-1} x_{j+1} X_j - \frac{1}{2}\sum_{j\neq n} \mathbb{N}_j^2 - x_{n+1}X_{n-1} - \beta_n X_{n-1} + \gamma_n x_{n+1} - \frac{\alpha_n^2}{2}$$

$\mathbb{N}_j = 1 - x_j X_j$. The continuum limit immediately leads to the familiar NLS expressions. Extra consistency check!

The $\mathbb{A}$-operators:

$$\mathbb{A}_j^{(1)}(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\mathbb{A}_j^{(2)}$ for $j \neq n,\ \ n+1$ is given by

$$\mathbb{A}_j^{(2)}(\mu) = \begin{pmatrix} \mu & x_j \\ -X_{j-1} & 0 \end{pmatrix},$$

whereas

$$\mathbb{A}_n^{(2)} = \begin{pmatrix} \mu & \beta_n + x_{n+1} \\ -X_{n-1} & 0 \end{pmatrix}, \qquad \mathbb{A}_{n+1}^{(2)} = \begin{pmatrix} \mu & x_{n+1} \\ \gamma_n - X_{n-1} & 0 \end{pmatrix}.$$

The continuum limit leads to the familiar NLS expressions for the $\mathbb{V}$-operators.

The NLS model: Lax pair

For the left and right bulk theories, and the defect point:

$$\mathbb{V}^{\pm(1)}(\mu,\ x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbb{V}^{-(2)}(\mu,\ x) = \begin{pmatrix} \mu & \bar{\psi}^{-}(x) \\ \psi^{-}(x) & 0 \end{pmatrix}$$

$$\mathbb{V}^{+(2)}(\mu,\ x) = \begin{pmatrix} \mu & \bar{\psi}^{+}(x) \\ \psi^{+}(x) & 0 \end{pmatrix}$$

$$\tilde{\mathbb{V}}^{-(2)}(\mu,\ x_0) = \begin{pmatrix} \mu & \bar{\psi}^{+}(x_0) + \beta(x_0) \\ \psi^{-}(x_0) & 0 \end{pmatrix}$$

$$\tilde{\mathbb{V}}^{+(2)}(\mu,\ x_0) = \begin{pmatrix} \mu & \bar{\psi}^{+}(x_0) \\ \gamma(x_0) + \psi^{-}(x_0) & 0 \end{pmatrix}$$

Discussion

- Deeper understanding of the *off-shell* vs *on-shell* integrability; related to suitable continuum limits.
- Extend the study to other classical integrable models with defects e.g. (an)isotropic Heisenberg chains, sigma models, and higher rank generalizations.
- Study of extended (not point like) defects, and defects associated to *non-ultra-local* algebras. Investigate also non-dynamical defects.
- At the quantum level: derive the associated transmission amplitudes via the Bethe ansatz equations.