Exact symmetries associated to reflection algebras and (q) twisted Yangians

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Anastasia Doikou

INFN Bologna

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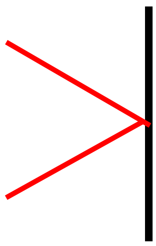
A.D. and N. Crampe, to appear.

SP b.c. ((q) reflection algebra)

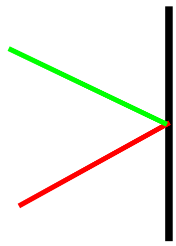


Spin chains:

(de Vega, Gonzalez Ruiz, A.D., Nepomechie...)

Integrable field Theory?

SNP b.c ((q) Twisted Yangian)



ATFT:

(Bowcock, Corrigan, Dorey, Rietdijk '95)

Spin chains:

(A.D. '00)

(Arnaudon, Avan, Crampe, A.D., Frappat, Ragoucy '04 '05)

Outline

(I) SP b.c., reflection algebra:

• Review R and K matrices as reps of the (affine) Hecke algebras. New rep of blob algebra (asymmetric twin). (Focus on trigon. case).

• Introduce RA and boundary quantum algebra $\rightarrow$ symmetry of transfer matrix. Symmetry of asymmetric twin chain.

• Duality: boundary quantum algebra and B type Hecke algebra.

(II) SNP b.c., (q) twisted Yangian:

• Review R and $\bar{R}$ and connection with (q) Brauer algebra. (Rational and trigonometric).

• Introduce (q) twisted Yangian and $\mathbb{BT} \rightarrow$ exact symmetry of the transfer matrix (rational). Indriguing results for q-case.

• Duality: (q) Brauer and $\mathbb{BT}$.

R matrices from the Hecke algebra

The R matrix acts on $\mathbb{V}^{\otimes 2}$, satisfies the YBE (Baxter '72)

$$R_{12}(\lambda_1 - \lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

The Hecke algebra $\mathcal{H}_N(q)$ *with* g_i, $i = 1, \ldots, N-1$*:*

$$(g_i - q)\ (g_i + q^{-1}) = 0$$
$$g_i\ g_{i+1}\ g_i = g_{i+1}\ g_i\ g_{i+1},$$
$$[g_i,\ g_j] = 0, \quad |i-j| > 1.$$

Let $\mathbb{U}_i = g_i - q$, Temperley–Lieb quotient:

$$\mathbb{U}_{i\pm1}\ \mathbb{U}_i\ \mathbb{U}_{i\pm1} = \mathbb{U}_{i\pm1}$$

Reps of $\mathcal{H}_N \longrightarrow$ solution YBE (Jimbo '86)

$$\check{R}_{i\ i+1}(\lambda) = \sinh(\lambda + i\mu)\ \mathbb{I} + \sinh\lambda\ \rho(\mathbb{U}_i).$$

Examples:

$\mathcal{U}(\widehat{gl_n})$: $(e_{ij})_{kl} = \delta_{ik}\ \delta_{jl}$, and U on $(\mathbb{C}^n)^{\otimes 2}$:

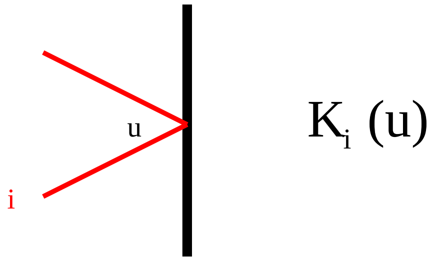
$$U = \sum_{i \neq j} (e_{ij} \otimes e_{ji} - q^{-sgn(i-j)} e_{ii} \otimes e_{jj}).$$

$\mathcal{U}_q(\widehat{sl_2})$: The XXZ model U on $(\mathbb{C}^2)^{\otimes 2}$

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Soliton preserving b.c./ Reflection algebras

The K matrix acts on $\mathbb{V}$:



Satisfies the reflection equation (Cherednik '84)

$$
\begin{aligned}
& R_{12}(\lambda_1 - \lambda_2)\ K_1(\lambda_1)\ R_{21}(\lambda_1 + \lambda_2)\ K_2(\lambda_2) \\
= & K_2(\lambda_2)\ R_{12}(\lambda_1 + \lambda_2)\ K_1(\lambda_1)\ R_{21}(\lambda_1 - \lambda_2)
\end{aligned}
$$

• Solutions of RE (e.g. via Hecke algebras: Levy and Martin '94, A.D. and Martin '02, A.D. '04) $\longrightarrow$ build open spin chains (Sklyanin '88)

The affine Hecke and blob algebra

The algebra $\mathcal{H}_N^0(q,Q)$, g_i, $i \in \{1,\ldots,N-1\}$ *and* g_0*:*

$$g_1 \; g_0 \; g_1 \; g_0 = g_0 \; g_1 \; g_0 \; g_1$$
$$[g_0, \; g_i] = 0, \quad i > 1$$

$\mathcal{H}_N^0(q,Q) \longrightarrow$ candidate solutions of RE (Levy and Martin '94).
Quotients of $\mathcal{H}_N^0$, B-type $\mathcal{B}_N(q,Q)$:

$$(g_0 - Q)(g_0 + Q^{-1}) = 0.$$

Blob algebra b_N extension of the Temperley–Lieb algebra, $\mathrm{e} = g_0 - Q$:

$$\mathbb{U}_1 \; \mathrm{e} \; \mathbb{U}_1 = \kappa \; \mathbb{U}_1$$

Reps of $\mathcal{B}_N(q,Q) \longrightarrow$ solutions to RE

$$K(\lambda) = x(\lambda)\mathbb{I} + y(\lambda)\rho(\mathrm{e}).$$

$U_b \longrightarrow$ rep of e on $\mathbb{C}^n$ (A.D. '04)

$$U_b = \frac{1}{2i\sinh i\mu}(-Q^{-1}\hat{e}_{11} - Q\hat{e}_{nn} + \hat{e}_{1n} + \hat{e}_{n1}).$$

(Blob case: Martin and Saleur '94). K matrix $\longrightarrow$ (Abad and Rios '95) for the $\mathcal{U}_q(\widehat{gl_n})$ case.

e.g. $\mathcal{U}_q(\widehat{sl_2})$

$$U_b = \begin{pmatrix} -Q^{-1} & 1 \\ 1 & -Q \end{pmatrix}$$

(rep of the blob algebra)

Twin representation

Let $\mathcal{R}_q : \mathcal{T}_N(q) \longrightarrow \text{End}((\mathbb{C}^2)^{\otimes N})$ the XXZ rep of $\mathcal{H}_N$:

$$\mathcal{R}_q(\mathbb{U}_l) = \mathbb{I} \otimes \ldots \otimes \underbrace{U}_{l,\ l+1} \otimes \ldots \otimes \mathbb{I}$$

then define 'twin' rep (Martin and Woodcock '98)

$\Theta_q : \mathcal{T}_N(q) \longrightarrow \text{End}((\mathbb{C}^4)^{\otimes N})$:

$$\Theta(\mathbb{U}_l) = \mathcal{R}_r(\mathbb{U}_{N-l})\ \mathcal{R}_{\hat{r}}(\mathbb{U}_{N+l})$$

$r\ \hat{r} = -q$.

Spin chain like model (Bethe ansatz and Hamiltonian symmetry) (A.D. and Martin '03, '05).

Define matrices in End$(\mathbb{V}^{\otimes 2N})$ acting non-trivially on $V_N \otimes V_{N+1}$:

$$\mathcal{M}^{i}(Q) = -\frac{\delta_e}{Q+Q^{-1}}\, \mathbb{I}\otimes\mathbb{I}\otimes\ldots\otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -Q & 1 & 0 \\ 0 & 1 & -Q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes\ldots\otimes\mathbb{I}\otimes\mathbb{I}$$

$$\mathcal{M}^{ii}(Q) = -\frac{\delta_e}{Q+Q^{-1}}\mathbb{I}\otimes\mathbb{I}\otimes\ldots\otimes \begin{pmatrix} -Q & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -Q^{-1} \end{pmatrix} \otimes\ldots\otimes\mathbb{I}\otimes\mathbb{I}$$

$$\mathcal{M}^{+}(Q) = \mathcal{M}^{i}(Q) + \mathcal{M}^{ii}(Q)$$

$$\begin{aligned}\mathcal{M}^{iii}(Q_1,Q_2) &= \frac{\delta_e}{(Q_1+Q_1^{-1})(Q_2+Q_2^{-1})}\,\mathbb{I}\otimes\ldots\otimes\begin{pmatrix} -Q_1 & 1 \\ 1 & -Q_1^{-1}\end{pmatrix} \\ &\otimes \begin{pmatrix} -Q_2^{-1} & 1 \\ 1 & -Q_2\end{pmatrix}\otimes\ldots\otimes\mathbb{I}\end{aligned}$$

$Q_1\ Q_2 = -Q$.

Representation $\Theta^I : b_N \longrightarrow \mathrm{End}(\mathbb{V}^{\otimes 2N})$

$$\Theta^I(\mathbb{U}_i) = \Theta(\mathbb{U}_i), \qquad \Theta^I(\mathrm{e}) = \mathcal{M}^I(Q)$$

provided

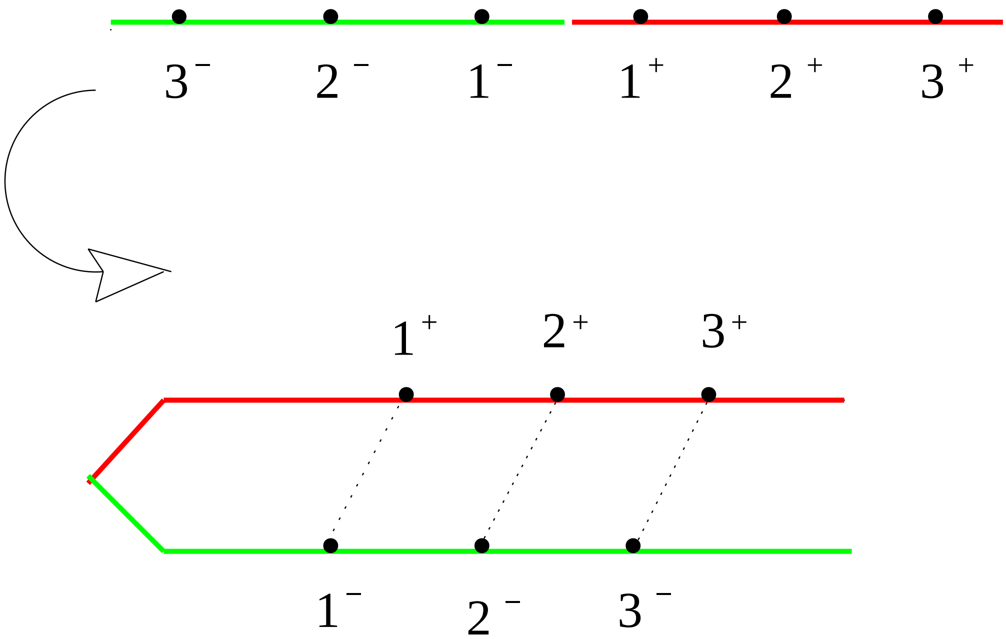
$$\delta_e = -\frac{Q + Q^{-1}}{2\mathrm{i}\sinh(\mathrm{i}\mu)}, \qquad \kappa^i = \frac{q^{-1}Q + qQ^{-1}}{2\mathrm{i}\sinh(\mathrm{i}\mu)}, \qquad \kappa^{ii} = \frac{\mathrm{i}Q - \mathrm{i}Q^{-1}}{2\mathrm{i}\sinh(\mathrm{i}\mu)}$$
$$\kappa^+ = \kappa^i + \kappa^{ii} \qquad \kappa^{iii} = \frac{q^{-1}Q + qQ^{-1} + 2}{2\mathrm{i}\sinh(\mathrm{i}\mu)}.$$

New class of reps of (boundary) Temperley–Lieb algebra:
Crossing reps (A.D. and Martin '05), $\rho^c : \; b_N \longrightarrow \mathrm{End}(\mathbb{C}^n)^{\otimes N}$

$$\rho^c(\mathbb{U}_i) = \mathcal{V}_{i+1}\, \mathcal{P}_{i\;i+1}^{t_i}\, \mathcal{V}_i, \quad i \in \{1, \ldots, N-1\}$$

$\mathcal{P} \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ permutation operator, and $(\mathcal{V}^t)^2\, \mathcal{V}^2 = \mathbb{I}$.

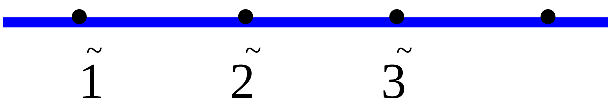
The folding



Setting

$$N+l \to l^+, \quad N-l+1 \to l^-, \quad l = 1, 2, \dots N$$

Composite index: $\tilde{l} = (l^-,\ l^-)$.



Reflection and boundary quantum algebras

$L(\lambda) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{A}$, ($\hat{L} = L^{-1}(-\lambda)$),

$R_{12}(\lambda_1 - \lambda_2)\ L_{13}(\lambda_1)\ L_{23}(\lambda_2) = L_{23}(\lambda_2)\ L_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$.

then

$$\mathbb{K}(\lambda' \mp \lambda) = L(\lambda' \mp \lambda)\ (K(\lambda') \otimes \mathbb{I})\ \hat{L}(\lambda' \pm \lambda)$$

$\mathbb{K}$ generates $\mathbb{R}$ (Sklyanin '88).

$$\Delta : \mathbb{R} \to \mathbb{R} \otimes \mathcal{A}, \quad \Delta(\mathbb{K}_{ab}(\lambda)) = \sum_{k,l=1}^{n} \mathbb{K}_{kl}(\lambda) \otimes R_{ak}(\lambda)\ \hat{L}_{lb}(\lambda).$$

RE as $\lambda' \to \infty$

$$R_{12}^{\pm(\mp)}\ \mathbb{K}_1^{\pm}\ \hat{R}_{12}^{\pm}\ \mathbb{K}_2^{\pm} = \mathbb{K}_2^{\pm}\ R_{12}^{\pm}\ \mathbb{K}_1^{\pm}\ \hat{R}_{12}^{\pm(\mp)}$$

$\mathbb{K}^{\pm}$ generate boundary quantum algebra $\mathbb{B}$.

The open spin chain

Integrable boundary conditions (Sklyanin '88)

$$\mathcal{T}(\lambda) = L_{0N}(\lambda) \dots L_{01}(\lambda)\ K^{(R)}(\lambda)\ \hat{L}_{01}(\lambda) \dots L_{0N}(\lambda)$$

$\mathcal{T}_{ab} = \Delta^{(N)}(\mathbb{K}_{ab})$.

$$t(\lambda) = tr_0 \left\{ K_0^{(L)}(\lambda)\ \mathcal{T}(\lambda) \right\}$$

$$\Big[t(\lambda),\ t(\lambda')\Big] = 0$$

Integrability ensured.
Boundary symmetries (A.D. '04, '05, '06).

Symmetry

$K^{(L)} = \mathbb{I}$, $K^{(R)}(\lambda)$ any solution of RE

$$R_{12}(\lambda)\ R_{21}(-\lambda) \propto \mathbb{I}, \qquad M_1^{-1}\ R_{12}^{t_1}(\lambda)\ M_1\ R_{21}^{t_1}(-\lambda - 2\mathrm{i}\rho) \propto \mathbb{I}$$

$$M = M^t, \qquad \Big[M_1\ M_2,\ R_{12}(\lambda)\Big] = 0,$$

$$\tau^{\pm} = tr_0\Big\{\mathbb{P}_0\ \mathcal{T}_0^{\pm}\Big\}, \qquad \mathcal{T}^{\pm} \propto \mathcal{T}(\lambda \to \pm\infty),$$

$\mathbb{P}$ $n \times n$ matrix.

$$\Big[\tau^{\pm},\ t(\lambda)\Big] = 0.$$

For $\mathbb{P} = E_{ab}$

$$\tau^{\pm} = \mathcal{T}_{ab}^{\pm} \quad \Rightarrow \quad \Big[\mathcal{T}_{ab}^{\pm},\ t(\lambda)\Big] = 0,$$

But $\mathcal{T}_{ab}^{\pm} = \Delta^{(N)}(\mathbb{K}_{ab}^{\pm}) \to \mathbb{B}$ provides a *symmetry* of $t(\lambda)$ for SP b.c. (A.D. '06). If $K^{(L)} \neq \mathbb{I}$ reduced symmetry.

Centralizer

Restrict $L \to R$, recall, $\check{R} = \mathcal{P}\ R$,
$\rho : \mathcal{B}_N \to \mathrm{End}(\mathbb{V}^{\otimes N})$, $\lambda \to \pm\infty$:

$$\check{R}^{\pm}_{i\ i+1} \propto \rho(\mathbb{U}_i) + q^{\pm 1}, \qquad K^{\pm}_1 \propto \rho(\mathrm{e}) + \mathrm{Q}^{\pm 1}$$

YBE $\lambda_{1,2} \to \pm\infty$:

$$\check{R}^{\pm}_{i\ i+1}\ R^{\pm}_{0\ i+1}\ R^{\pm}_{0i} = R^{\pm}_{0\ i+1}\ R^{\pm}_{0i}\ \check{R}^{\pm}_{i\ i+1},$$
$$\check{R}^{\pm}_{i\ i+1}\ R^{\pm}_{i0}\ R^{\pm}_{i+1\ 0} = R^{\pm}_{i0}\ R^{\pm}_{i+1\ 0}\ \check{R}^{\pm}_{i\ i+1}$$

$$\Big[\rho(\mathbb{U}_i),\ \mathcal{T}^{\pm}_{ab}\Big] = 0, \qquad \Big[\rho(e),\ \mathcal{T}^{\pm}_{ab}\Big] = 0 \qquad i \in \{1, \dots, N-1\}.$$

Duality between $\mathcal{B}_N$ and $\mathbb{B}$ (A.D. '06).

Rational case follows along the same lines ($q \to 1$) (A.D. '04).

Examples:

$\mathcal{U}_q(\widehat{gl_n})$: symmetry $K^{(L)} \propto \mathbb{I}$, $K^{(R)} \neq \mathbb{I}$ non-local charges identified (A.D. '05) ($K^{(L,R)} \propto \mathbb{I}$) $\mathbb{B}$ coincides with $\mathcal{U}_q(gl_n)$.

$\mathcal{U}_q(\widehat{sl_2})$: symmetry $K^{(L)} \propto \mathbb{I}$, $K^{(R)} \neq \mathbb{I}$

$$\mathcal{T}_{11}^{+} = xq^{2S^z} + q^{-\frac{1}{2}}q^{S^z}S^{+} + q^{-\frac{1}{2}}q^{S^z}S^{-}$$

(Mezincescu and Nepomechie '97, Delius and Mackay '03)

Symmetry and centralizer (A.D. '04)

$$\Big[t(\lambda),\ \mathcal{T}_{11}^{+}\Big] = 0.$$

let $\pi : \mathcal{U}_q(sl_2) \to \mathbb{C}^2$ then

$$\Big[U_l,\ \pi^{\otimes N}(\mathcal{T}_{11}^{+})\Big] = 0 \to \Big[\mathcal{H},\ \pi^{\otimes N}(\mathcal{T}_{11}^{+})\Big] = 0$$

Twin rep. Familiar non-local charges

$t(\lambda) \in (\mathbb{C}^4)^{\otimes N}$ ($q = e^{i\mu}$, $\mathrm{i} = \sqrt{-1}$). Introduce reps (A.D. and Martin '05):

$\sigma_1 : \mathcal{U}_{\mathrm{i}}(sl_2) \longrightarrow$ End$(\mathbb{C}^2 \otimes \mathbb{C}^2)$ and $\sigma_2 : \mathcal{U}_q(sl_2) \longrightarrow$ End$(\mathbb{C}^2 \otimes \mathbb{C}^2)$

$$\sigma_1(\mathcal{H}) = \mathrm{i}^{\frac{1}{2}(e_{11}-e_{44})}, \quad \sigma_1(\mathcal{E}) = e_{14}, \quad \sigma_1(\mathcal{F}) = e_{41}$$
$$\sigma_2(\mathcal{H}) = q^{-\frac{1}{2}(e_{22}-e_{33})}, \quad \sigma_2(\mathcal{E}) = e_{32}, \quad \sigma_2(\mathcal{F}) = e_{23}$$

$\rho_1 : \mathcal{U}_{\hat{r}}(sl_2) \longrightarrow$ End$(\mathbb{C}^2 \otimes \mathbb{C}^2)$ and $\rho_2 : \mathcal{U}_r(sl_2) :\longrightarrow$ End$(\mathbb{C}^2 \otimes \mathbb{C}^2)$

$$\rho_1(\mathcal{H}) = \mathbb{I} \otimes \hat{r}^{\frac{1}{2}\sigma^z}, \quad \rho_1(\mathcal{E}) = \mathbb{I} \otimes \ \sigma^+, \quad \rho_1(\mathcal{F}) = \mathbb{I} \otimes \ \sigma^-$$
$$\rho_2(\mathcal{H}) = r^{-\frac{1}{2}\sigma^z} \otimes \mathbb{I}, \quad \rho_2(\mathcal{F}) = \sigma^+ \otimes \mathbb{I}, \quad \rho_2(\mathcal{E}) = \sigma^- \otimes \mathbb{I}.$$

Symmetry: (A.D '06)

$$(\mathrm{h} \otimes \mathrm{h}^{\otimes N})\Delta'^{(N+1)}(x)\ \mathcal{T}(\lambda) = \mathcal{T}(\lambda)\ (\mathrm{h} \otimes \mathrm{h}^{\otimes N})\Delta'^{(N+1)}(x).$$

$x \in \mathcal{U}_{\mathrm{q}}(sl_2)$, $h \in \{\sigma_i,\ \rho_i\}$, $\mathrm{q} \in \{q, \imath,\ r,\ \hat{r}\}$
$K^{(L)} = K^{(R)} = \mathbb{I}$ trivial:

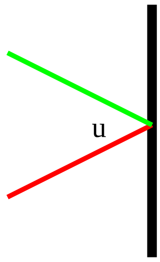
$$\Big[t(\lambda),\ \mathcal{U}_q(sl_2)\Big] = \Big[t(\lambda),\ \mathcal{U}_{\mathrm{i}}(sl_2)\Big] = 0$$

$$\Big[t(\lambda),\ \mathcal{U}_r(sl_2)\Big] = \Big[t(\lambda),\ \mathcal{U}_{\hat{r}}(sl_2)\Big] = 0$$

For $K^{(R)} \neq \mathbb{I}$ non-local charges as in XXZ. Intriguing task: relation 'familiar' charges with $\mathcal{T}_{ii}^{\pm} \rightarrow$ relation with spectrum (for XXZ, A.D. '06)!
Diagonalization of charges important (for XXZ, Nichols, Rittenberg, de Gier '05, Baseilhac '06)

Soliton non preserving b.c./(q) twisted Yangian

Define: $\bar{R}(\lambda) = R_{12}^{t_1}(-\lambda - i\rho)$. The $\mathcal{K}$ matrix



Satisfies the (q) twisted Yangian

$$R_{12}(\lambda_1 - \lambda_2)\ \mathcal{K}_1(\lambda_1)\ \bar{R}_{12}(\lambda_1 + \lambda_2)\ \mathcal{K}_2(\lambda_2) = \mathcal{K}_2(\lambda_2)\ \bar{R}_{12}(\lambda_1 + \lambda_2)\ \mathcal{K}_1(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

The spin chain-like model

$\bar{L}_{12}(\lambda) = L_{12}^{t_1}(-\lambda - i\rho)$

$$\mathbb{K}(\lambda' \mp \lambda) = L(\lambda' \mp \lambda)\ (\mathcal{K}(\lambda') \otimes \mathbb{I})\ \bar{L}(\lambda' \pm \lambda).$$

$\mathbb{K}$ generate $\mathbb{T}$, endowed with a coporduct $\Delta : \mathbb{T} \to \mathbb{T} \otimes \mathcal{A}$

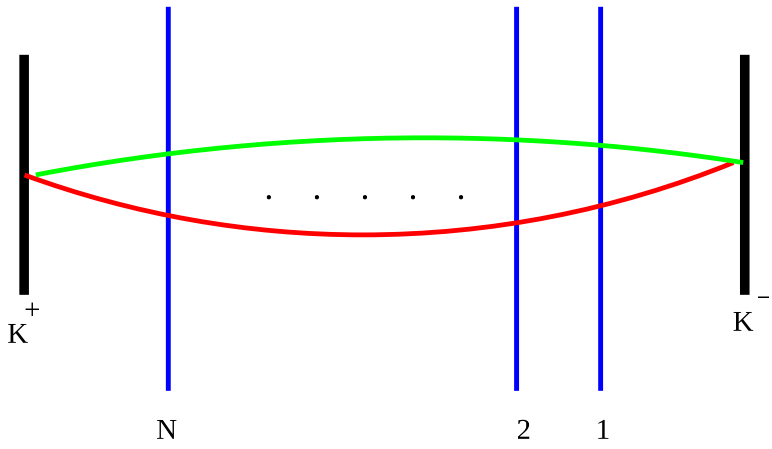
(Delius and Mackay'03)

$$\Delta(\mathbb{K}_{ab}(\lambda)) = \sum_{k,l=1}^{n} \mathbb{K}_{kl}(\lambda) \otimes L_{ak}(\lambda)\bar{L}_{lb}(\lambda) \qquad a,\ b \in \{1, \dots, n\}.$$

$$\mathcal{T}_0(\lambda) = L(\lambda) = L_{0N}(\lambda) \dots L_{01}(\lambda)\ \mathcal{K}_0^{(R)}(\lambda)\ \bar{L}_{01}(\lambda) \dots \bar{L}_{0N}(\lambda),$$

$\mathcal{T}_{ab}(\lambda) = \Delta^{(N)}(\mathbb{K}_{ab}(\lambda))$.

$$t(\lambda) = Tr_0 \left\{ \mathcal{K}_0^{(L)}(\lambda)\ \mathcal{T}_0(\lambda) \right\},$$



$$\Big[t(\lambda),\ t(\lambda') \Big] = 0.$$

Spin chain built (BAE and symmetry sl_3) (A.D. '00), generalized (Arnaudon, Avan, Crampe, A.D., Frappat, Ragoucy '04, '05)

The rational case

$$R(\lambda) = \mathbb{I} + \frac{i}{\lambda}\mathcal{P}, \quad \bar{R}(\lambda) = \mathbb{I} + \frac{i}{\lambda}\check{\mathcal{P}}$$

$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$
$\mathcal{P} = \sum_{a,b} E_{ab} \otimes E_{ba}$, $\check{P} = \rho - \mathcal{Q}$, $\rho = \frac{n}{2}$ and $\mathcal{Q} = \sum_{a,b} E_{ab} \otimes E_{ab}$ one dim projector

$$\mathcal{Q}\ \mathcal{P} = \mathcal{P}\ \mathcal{Q} = \pm\mathcal{Q}, \qquad \mathcal{Q}^2 = n\mathcal{Q},$$

and $\check{\mathcal{P}}^2 = \rho^2\mathbb{I}$.

Brauer algebra $\mathrm{B}_N(\delta)$ $2N-2$ generators g_i, τ_i:

$$\begin{aligned} g_i^2 &= 1, \quad \tau_i^2 = \delta\ \tau_i, \quad g_i\ \tau_i = \tau_i\ g_i = \tau_i, \quad i = 1,\ldots,N \\ g_i\ g_j &= g_j\ g_i, \quad \tau_i\ \tau_j = \tau_j\ \tau_i, \quad g_i\ \tau_j = \tau_j\ g_i, \quad |i-j| > 1 \\ g_i\ g_{i+1}\ g_i &= g_{i+1}\ g_i\ g_{i+1}, \quad \tau_i\ \tau_{i\pm1}\ \tau_i = \tau_i, \\ g_i\tau_{i+1}\ \tau_i &= g_{i+1}\ \tau_i, \quad \tau_{i+1}\ \tau_i\ g_{i+1} = \tau_{i+1}\ g_i, \quad i = 1,\ldots N-2 \end{aligned}$$

$\mathcal{P}$ and $\mathcal{Q}$ form rep of $\mathrm{B}_N(\delta)$, $g_i \to \mathcal{P}_{i\ i+1}, \quad \tau_i \to \mathcal{Q}_{i\ i+1}$

Symmetry

$$L(\lambda) = \mathbb{I} + \frac{i}{\lambda}\mathbb{P}, \qquad \bar{L}(\lambda) = L^{t_1}(-\lambda - i\rho) = \mathbb{I} + \frac{i}{\lambda}\check{\mathbb{P}}$$

$\mathbb{P}_{ab} \in gl(n),\ n \in \{1, \ldots, n\}$ and $\check{\mathbb{P}}_{ab} = \rho - \mathbb{P}_{ba}$. $\mathcal{P}_{ab},\ \check{\mathcal{P}}_{ab}$ fundamental reps of $\mathbb{P}_{ab},\ \check{\mathbb{P}}_{ab}$.

$\lambda \to \infty$, i.e.

$$\mathcal{T}(\lambda \to \infty) \propto \mathcal{K} + \frac{i}{\lambda}\mathcal{T}^{(0)} + \ldots.$$

From (q) twisted Yangian

$$\Big((\mathcal{P}_{ac}\,\mathcal{K}_{cb} + \mathcal{K}_{ac}\,\check{\mathcal{P}}_{cb}) \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{T}^{(0)}_{ab}\Big)\ \mathcal{T}(\lambda) = \mathcal{T}(\lambda)\ \Big((\check{\mathcal{P}}_{ac}\,\mathcal{K}_{cb} + \mathcal{K}_{ac}\,\mathcal{P}_{cb}) \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{T}^{(0)}_{ab}\Big)$$

$$\Big[t(\lambda),\ \mathcal{T}^{(0)}_{ab}\Big] = 0.$$

(Crampe and A.D. '06)

• If $\mathcal{K}^{(L,\ R)} = \mathbb{I}$ or $\mathcal{K}^{(L,\ R)} = antidiag(1, 1, ..., 1)$ then $\mathcal{T}_{ab}^{(0)}$ form the $so(n)$ algebra,

• If $\mathcal{K}^{(L,\ R)} = antidiag(i, -i, ..., -i)$ $\ n$ even only, then $\mathcal{T}_{ab}^{(0)}$ form the $sp(n)$ algebra.

Centralizer:
The fund reps of $\mathcal{T}^{(0)}$ not centralizer of Brauer. But restrict

$$\mathcal{O}_{ab} = \mathcal{K}_{ac}\check{\mathcal{P}}_{cb} + \mathcal{P}_{ac}\mathcal{K}_{cb}, \qquad \bar{\mathcal{O}}_{ab} = \mathcal{K}_{ac}\mathcal{P}_{cb} + \check{\mathcal{P}}_{ac}\mathcal{K}_{cb}$$

$\mathcal{W}_{ab} = \mathcal{O}_{ab} + \bar{\mathcal{O}}_{ab}$ invariant $\mathcal{P}_{ab} \to \check{\mathcal{P}}_{ab}$.

$$\mathcal{P}\ \Delta(\mathcal{W}_{ab}) = \Delta(\mathcal{W}_{ab})\ \mathcal{P}, \qquad \mathcal{Q}\ \Delta(\mathcal{W}_{ab}) = \Delta(\mathcal{W}_{ab})\ \mathcal{Q}.$$

Recall $\mathcal{P},\ \mathcal{Q}$ rep of Brauer $\to$ $\Delta^{(N)}(\mathcal{W}_{ab})$ centralizer of the Brauer B_N (Crampe and A.D. '06).

Boundary extension?

Restricting to $\mathcal{K} = \mathcal{K}^{-1}$ possible boundary extension. Intro b, satisfying

$$\mathrm{b}^2 = 1, \qquad g_1 \ \mathrm{b} \ \tau_1 \ \mathrm{b} = \mathrm{b} \ \tau_1 \ \mathrm{b} \ g_1.$$

More plausible scenario: $\mathcal{K} = \mathcal{K}^t$ let

$$\mathcal{Q}'_{i\ i+1} = \mathcal{K}_i \mathcal{P}^{t_i}_{i\ i+1} \mathcal{K}_i^{-1} = \mathcal{K}_{i+1} \mathcal{P}^{t_i}_{i\ i+1} \mathcal{K}_{i+1}^{-1}$$

new representation of B_N:

$$g_i \to \mathcal{P}_{i\ i+1}, \quad \tau_i \to \mathcal{Q}'_{i\ i+1}.$$

$$\mathcal{P} \ \Delta(\mathcal{O}_{ab}) = \Delta(\mathcal{O}_{ab}) \ \mathcal{P}, \qquad \mathcal{Q}' \ \Delta(\mathcal{O}_{ab}) = \Delta(\mathcal{O}_{ab}) \ \mathcal{Q}'.$$

$\Delta^{(N)}(\mathcal{O}_{ab})$ center B_N (Crampe and A.D. '06).

The trigonometric case

Recall $R \longrightarrow$ Hecke algebra gens. $\bar{R}_{12}(-i\rho) = \mathcal{P}_{12}^{t_1}$.

Quantum Brauer $\mathrm{B}_N(z,q)$ (Molev '02) $g_1, \dots, g_N, \tau_{N-1}$

$$\begin{aligned}
g_i^2 &= (q - q^{-1}) g_i + 1, \qquad g_i\ g_j = g_j\ g_i, \qquad g_i\ g_{i+1}\ g_i = g_{i+1}\ g_i\ g_{i+1}, \\
\tau_k^2 &= \frac{z - z^{-1}}{q - q^{-1}} \tau_k, \qquad g_k\ \tau_k = \tau_k\ g_k = q \tau_k \\
\tau_k\ g_{k-1}\ \tau_k &= z\ \tau_k, \qquad g_i\ \tau_k = \tau_k\ g_i, \qquad i = 1, \dots k-2 \\
& \tau_k\ (zq\ \zeta^{-1} + z^{-1} q^{-1} \zeta)\ \tau_k\ (q \zeta^{-1} + q^{-1} \zeta) = \\
& (q\ \zeta^{-1} + q^{-1} \zeta)\ \tau_k\ (z\ q\ \zeta^{-1} + z^{-1} q^{-1} \zeta)\ \tau_k
\end{aligned}$$

where $k = N-1$ and $\zeta = g_{k-1}\ g_{k-2}\ g_k\ g_{k-1}$. g_i generators of $\mathcal{H}_N(q)$.

Rep: $g_i \longrightarrow U_{i\ i+1} + q, \qquad \tau_{N-1} \longrightarrow \mathcal{P}_{N-1\ N}^{t_{N-1}}\ M_{N-1}^{-1}$.

The $\mathcal{K}$ matrices

Classification in: (Arnaudon, Crampe, A.D., Frappat Ragoucy '05)

(i) $\mathcal{K}$ constant diagonal

(ii) $\mathcal{K}(\lambda) = \frac{q^2 - e^{4\mu(\lambda + i\frac{\rho}{2})}}{q+1} \sum_{i=1}^{n} E_{ii} - e^{2\mu\lambda}(e^{2\mu(\lambda + i\frac{\rho}{2})} \pm q) \sum_{i<j} E_{ij} + (q \pm e^{2\mu(\lambda + i\frac{\rho}{2})})) \sum_{i>j} E_{ij}$

(iii) $\mathcal{K}(\lambda) = e^{2\mu(\lambda + i\frac{\rho}{2})} E_{1n} - q e^{2\mu(\lambda + i\frac{\rho}{2})} E_{n1} + \sum_{i=1}^{\frac{n-2}{2}} (q E_{2i\ 2i+1} - E_{2i+1\ 2i}), \quad n$ even.

(iv) $\mathcal{K}(\lambda) = \sum_{i=1}^{\frac{n}{2}} (q E_{2i-1\ 2i} - E_{2i\ 2i-1}), \quad n$ even.

(i), (ii) coincide with (Gandenberger '99, Delius and Mackay '03)

Symmetry?

$\lambda \to \infty$ obtain $\mathcal{T}_{ab}^{\pm}$
$\mathcal{T}_{a\ a+1}^{+}$ for (ii) coincide (Delius and Mackay '03)
$\mathcal{T}_{ab}^{\pm}$ form $\mathbb{BT}$:

$$R_{12}^{\pm(\mp)}\ \mathcal{T}_1^{\pm}\ \bar{R}_{12}^{\pm}\ \mathcal{T}_2^{\pm} = \mathcal{T}_2^{\pm}\ \bar{R}_{12}^{\pm}\ \mathcal{T}_1^{\pm}\ R_{12}^{\pm(\mp)}$$

Use defining relations $\mathbb{BT} \to no$ symmetry formed by $\mathcal{T}_{ab}^{\pm}$ contrary to the rational case and the SP!

Centralizer:

Special case $\mathcal{K}^{(L,R)} \propto \mathbb{I}$. Let $\pi : \mathcal{U}_q(gl_n) \to \mathbb{C}^n$

$$\Big[\pi^{\otimes N}(\mathcal{T}_{ab}^{\pm}),\ \mathcal{P}_i\Big] = \big[\pi^{\otimes N}(\mathcal{T}_{ab}^{\pm}),\ \mathcal{Q}_i\Big] = 0.$$

They form $\mathcal{U}_q'(so_n)$ $(\mathcal{U}_q'(sp_n))$ (Molev '02)

Summary

(I) SP b.c., reflection algebra:

• $\mathbb{B}$ symmetry algebra for the open transfer matrix.

• Duality between $\mathbb{B}$ and $\mathcal{B}_N$ established for any right boundary. (Valid results for rational and trigo).

(II) SNP b.c., (q) twisted Yangian:

• Rational case: $\mathbb{BT}$ symmetry algebra for open spin chain (special left right boundaries).

• Duality between $\mathbb{BT}$ and Brauer via appr. reps of Brauer.

• Duality: (q) Brauer and $\mathbb{BT}$ only for trivial b.c. Symmetry?

Conclusions

• Possible boundary extension of the (q) Brauer algebra.

• Other reps of the (q) Brauer algebra commuting with boundary non-local charges. Establish duality for any boundary.

• Symmetry of the finite chain (trigonometric) with SNP B.C. Search for a non-abelian symmetry.

• Field theoretical results based on the reflection algebra.