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DEGENERACY OF STRING STATES IN 2-D BLACK HOLE AND A NEW DERIVATION OF SU(1,1) PARAFERMION CHARACTERS * †

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ABSTRACT

An explicit formula for the degeneracy of states of the two dimensional target space-time black hole is obtained by projecting to the Virasoro primary states of the coset model $SU(1,1)_{-k}/U(1)_{-k}$. As a byproduct we rederive the $SU(1,1)$ parafermion characters using group theoretic methods.

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1-Introduction.

During the last couple of years there has been considerable interest in the representation theory of the $SU(1,1)/U(1)$ coset model for several reasons. 1). Little is known about coset models based on non-compact groups [1] and the coset $SU(1,1)/U(1)$ is the simplest one to begin their study [2,1] . 2). To construct heterotic string theories based on $SU(1,1)$ and other non-compact groups and make phenomenological predictions the characters and their modular invariants are needed. 3). Bars and Nemeschansky introduced [3] a formulation of strings propagating in curved space-time in terms of current algebra anti-de- Sitter coset models $SO(d-1,2)_{-k}/SO(d-1,1)_{-k}$ which are equivalent to gauged WZWN models, where $d \geq 2$ is interpreted as the space time dimension. Recently, Witten showed [4] that the lowest dimensional of their models, corresponding to $d=2$, has an interpretation as a two-dimensional space-time black hole that has an interesting connection to the $c=1$ matrix models .

The purpose of this paper is to derive an expression which gives the degeneracy of the states of the two-dimensional black hole spectrum. This is done in two steps. First, in section 2, we present a new derivation of the characters of the $SU(1,1)_{-k}/U(1)_{-k}$ parafermions [5] using group theoretic methods. Our results are in agreement with those in ref. [6] in which the Feigin-Fuchs Coulomb gas method was used[1]. The black hole string states correspond to the $SU(1,1)/U(1)$ states further moded out by the Virasoro constraints. By using these character formulas and restricting, in the spirit of [3, 7], to only Virasoro primary states of the $SU(1,1)_{-k}/U(1)_{-k}$ theory we find the degeneracy formula in section 3.

2-The characters of the $SU(1,1)$ parafermions.

In order to obtain the parafermion characters we have to project $SU(1,1)_{-k}$ characters to sectors with specified $U(1)$ charge. It is a well-known fact that among all possible states in the $SU(1,1)_{-k}$ theory there are states with zero or negative norm . It has been shown [1,2] that if we project out the $U(1)$ subgroup, the remaining states generally are of positive norm if the condition $j+1 < k/2$ for the discrete representations is fulfilled and for no restriction for the continuous ones. However, on special conditions there are zero norm states [2, 7]. Therefore we start with the $SU(1,1)_{-k}$ characters (which include negative or zero norm states) that are easier to construct.

The $SU(1,1)_{-k}$ current algebra and the central charge of the Virasoro subalgebra is

[1] In fact, these characters were known to the authors of ref. [2],[10].

$$[J_n^0, J_m^\pm] = \pm J_{n+m}^\pm, \qquad [J_n^0, J_m^0] = -nk/2\delta_{n+m}$$
$$[J_n^+, J_m^-] = -2J_{n+m}^0 + nk\delta_{n+m}, \qquad c = \frac{3k}{k-2}, \qquad k > 2. \tag{2.1a}$$

The Fourier components of the Virasoro generators associated with $G = SU(1,1)_{-k}$ and $U(1)_{-k}$ are

$$L_n^G = \frac{1}{k-2} \sum_{m \in Z} : \left(-J_{n+m}^0 J_{-m}^0 + \frac{1}{2} J_{n+m}^+ J_{-m}^- + \frac{1}{2} J_{n+m}^- J_{-m}^+\right) :$$
$$L_n^H = -\frac{1}{k} \sum_{m \in Z} : J_{n+m}^0 J_{-m}^0 : . \tag{2.1b}$$

On any zero level state labelled as $|j, m\rangle_0$, the Lie-algebra operators act as

$$J_0^0 |j, m\rangle_0 = m|j, m\rangle_0, \qquad J_0^\pm |j, m\rangle_0 = \sqrt{m(m \pm 1) - j(j+1)}|j, m \pm 1\rangle_0. \tag{2.2}$$

The unitary representations of $SU(1,1)$ Lie-algebra are [8]
(1)-The singlet: $j = -1, \ m = 0$.
(2)-The principal series, $j = -\frac{1}{2} + i\frac{\rho}{2}$, $m = m_0 \pm n$; with $-\infty < \rho < \infty$, $-\frac{1}{2} \leq m_0 \leq \frac{1}{2}$, $n = 0, 1, 2, \cdots$.
(3)-The supplementary series, $j = -\frac{1}{2} + \frac{\sigma}{2}$, $m = m_0 \pm n$; with $0 < |\sigma| < 1$, and $-\frac{1}{2} + \frac{|\sigma|}{2} < |m_0| < \frac{1}{2} - \frac{|\sigma|}{2}$, $n = 0, 1, 2, \cdots$. Note that the end points $|m_0| = \frac{1}{2} - \frac{|\sigma|}{2}$ are not included since they belong to the discrete series. Furthermore, the points $\sigma = 0, \pm 1$ are excluded.
(4)-The positive discrete series (D^+), $j + 1 > 0$, real and $m = j+1, j+2, j+3, \cdots$.
(5)-The negative discrete series (D^-), $j+1 > 0$, real and $m = -(j+1), -(j+2), -(j+3), \cdots$.

In the presence of negative norm or null states, the most general state of the $SU(1,1)_{-k}$ current algebra is of the form

$$|\Psi_{j,m}; N_i(n_i)\rangle_l = \prod_{i=1}^{3} \prod_{n_i=1}^{\infty} \left(J_{-n_1}^0\right)^{N_1(n_1)} \left(J_{-n_2}^+\right)^{N_2(n_2)} \left(J_{-n_3}^-\right)^{N_3(n_3)} |j, m'\rangle_0. \tag{2.3}$$

The level l in which it appears and the eigenvalue of J_0^0, m are given by

$$l = \sum_{i=1}^{3} \sum_{n_i=1}^{\infty} n_i N_i(n_i), \qquad m = m' + \sum_{n_2=1}^{\infty} N_2(n_2) - \sum_{n_3=1}^{\infty} N_3(n_3) \tag{2.4}$$

Before we proceed to count the states and derive a degeneracy formula for general l and m let us consider for simplicity, the case when $l=1$. Generically we have three states

$$
\begin{aligned}
&|\Psi_{j,m}; N_2(1)=1\rangle_1 = J^+_{-1}|j, m'=m-1\rangle_0 \\
&|\Psi_{j,m}; N_1(1)=1\rangle_1 = J^0_{-1}|j, m'=m\rangle_0 \\
&|\Psi_{j,m}; N_3(1)=1\rangle_1 = J^-_{-1}|j, m'=m+1\rangle_0 .
\end{aligned}
\tag{2.5}
$$

Consider first states which correspond to a continuous series, either the principal or the supplementary one. Since m' can take values in the whole real line all three states in (2.5) are acceptable as states of the $SU(1,1)_{-k}$ current algebra. In fact, it is obvious that the same is true at any level l. Therefore as far as counting of the states is concerned, the answer is the same as in the case of three free bosons and it does not depend on m at all. Let $P_r(l)$ be the number of states at level l, in a theory of r free bosons, defined through the following expansion

$$
\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^r} = \sum_{l=0}^{\infty} P_r(l) q^l . \tag{2.6}
$$

The character formula for the $SU(1,1)$ module in the continuous representations is simply that of three free bosons

$$
\hat{\chi}_j = \frac{q^{-\frac{c}{24}+\Delta_j}}{\prod_{m=1}^{\infty}(1-q^m)^3}, \tag{2.7}
$$

where $\Delta_j = -\frac{j(j+1)}{k-2}$ is the conformal dimension of a $SU(1,1)_{-k}$ zero level primary field. Let us now turn to the states corresponding to the D^+ series. As long as $m' \geq j+1$, the corresponding state in (2.5) belongs to the module. Therefore for $m \geq j+2$ we should include all three states at level 1. For $m = j+1$ we have two states at level 1 because the first of the states in (2.5) corresponding to $m' = j$ should not be included in the base. For $m = j$ the second of the states in (2.5) is also omitted leaving us with only one state. Finally for $m \leq j-1$ there are no states at all at level 1. Generalizing this idea to higher levels l it should be clear by now that, for values of $m \geq j+1+l$ the counting of states is the same as in a theory of three free bosons and it does not depend on m just like the continuous series. As we lower the value of m we start omitting states because they are based on values of m' that are not included in the D^+ representation at the base, until we reach the value $m = j-l$ for which there will be no states at all. Finally we can obtain the results for the D^- series by simply reversing the value of m.

After this digression we return to the general problem and restrict ourselves to states corresponding to the D^+ series. Let's define a new quantum number through the relation

$m = j+1+n$. Thus, n takes values $n = -l, -l+1, \ldots$. For given m (or equivalently n) and l the contribution from states with $m' \leq j$ that are absent from D^+ have to be subtracted. Hence the states satisfying the first of eqs. (2.4) and

$$j - m' = \sum_{n_2=1}^{\infty} N_2(n_2) - \sum_{n_3=1}^{\infty} N_3(n_3) - 1 - p = 0 \qquad p = n, n+1, \cdots, l-1 \tag{2.8}$$

should be subtracted from the module. One finds that for $p \geq 2l$ there is no solution to that system of equations which implies that for $m \geq j+1+l$ (or $p \geq l$) the counting is like 3 free bosons.

Let us call the number of states satisfying (2.4,2.8) $A_{l,p}$ and the degeneracy number $d_3(l,n)$. Then $A_{l,p}$ and $d_3(l,n)$ are given by means of the following formulas

$$\begin{aligned} A_{l,p} &= \sum_{i=1}^{3} \sum_{N_i(n_i)=0}^{\infty} \delta(l, \sum n_j N_j(n_j))\ \delta(1+p, \sum_{n_2} N_2(n_2) - \sum_{n_3} N_3(n_3)) \\ d_3(l,n) &= P_3(l) - \sum_{p=n}^{l-1} A_{l,p}, \end{aligned} \tag{2.9}$$

where by $\delta(n,m)$ we denote the usual Cronecker δ-symbol. The character $\hat{\chi}_j^+$ corresponding to the D^+ series is defined as follows

$$\hat{\chi}_j^+(q,\theta) = q^{-\frac{c}{24}+\Delta_j} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} d_3(l,n) q^l e^{-i(j+1+n)\theta}, \tag{2.10}$$

Using (2.6) for $r=3$, (2.8,9) and the following two identities

$$A_{l,-l-1} = 1, \qquad \sum_{p=-l-1}^{l-1} A_{l,p} = P_3(l) \tag{2.11}$$

we find after some algebraic manipulations that

$$\begin{aligned} \hat{\chi}_j^+(q,\theta) &= \frac{q^{-\frac{c}{24}+\Delta_j} e^{-i(j+1)\theta}}{\prod_{m=1}^{\infty}(1-q^m)(1-q^m e^{i\theta})(1-q^{(m-1)}e^{-i\theta})} \\ &= \frac{q^{-\frac{c}{24}+\Delta_j}}{\prod_{m=1}^{\infty}(1-q^m)^3} \sum_{n=-\infty}^{+\infty} \sum_{s=0}^{\infty} (-1)^s e^{-i(j+1+n)\theta} q^{\frac{1}{2}s(s+2n+1)}, \end{aligned} \tag{2.12}$$

where in order to write the second equality we have made use of well known formulas [9]. An interesting point is that, the first line in (2.12) is proportional to the inverse of the denominator of the $SU(2)_k$ character formula.

Using (2.10,12) we can easily project to sectors with definite $U(1)$ charge by taking the coefficient of $e^{-im\theta}$ and thus find the characters of the $SU(1,1)/U(1)$ parafermions. They read

$$\begin{aligned}
\chi^{\pm}_{j,n}(q) &= \frac{q^{-\frac{\tilde{c}}{24}+h_{j,n\pm(j+1)}}}{\prod_{k=1}^{\infty}(1-q^k)^2}\sum_{s=0}^{\infty}(-1)^s q^{\frac{1}{2}s(s+2n+1)} \qquad \text{Discrete,} \\
\chi_{j,m} &= \frac{q^{-\frac{\tilde{c}}{24}+h_{j,m}}}{\prod_{k=1}^{\infty}(1-q^k)^2} \qquad \text{Continuous,}
\end{aligned} \tag{2.13}$$

where in the first line $n \in Z$. For the continuous series $-\infty < m < \infty$. The ranges of j were given above for the various representations. The central charge and the conformal dimensions of the zero level primary fields are

$$\tilde{c} = c - 1 = \frac{2(k+1)}{k-2}, \qquad h_{j,m} = -\frac{j(j+1)}{k-2} + \frac{m^2}{k}. \tag{2.14}$$

The characters (2.13) generically contain only positive norm states as long as $j+1 < k/2$ for the discrete series (unless special conditions are imposed on the quantum numbers, in which case there are zero norms in both discrete or continuous series [1, 7]). There is no restriction for the continuous series. Notice also that, they are exactly the characters the authors of ref. [6] obtained following the Coulomb gas approach.

3-Degenaracy of states and two dimensional black hole.

String theories in curved space-time, formulated in terms of current algebra anti-de-Sitter coset models $SO(d-1,2)_{-k}/SO(d-1,1)_{-k}$, were first introduced in [3]. Recently, Witten showed [4] that, the lowest dimensional of these models corresponding to $d=2$ describes a black hole in a two-dimensional target space-time. In particular, when $k = 9/4$, which corresponds to $\tilde{c} = 26$ can be considered as a bosonic string background in its own right [3,4,7]. We consider the analytic continuation to the Euclidean black hole [7].

$$L_n^{G/H} = J_n^0 = 0, \ n \geq 1, \qquad L_0^{G/H} = 1, \tag{3.1}$$

where

$$L_0^{G/H} = -\frac{j(j+1)}{k-2} + \frac{m^2}{k} + l, \qquad l = 0, 1, 2 \cdots. \tag{3.2}$$

Excited states for $l \geq 2$ require discrete series with $j+1 > k/2$ which exceed the unitarity domain $j+1 < k/2$. However, only a few of the states in such modules satisfy the Virasoro projection of (3.1). For the remaining states, one may ask if there is a projection on a unitary subspace. Bars [7] proposed that for $k = 9/4$ a possible projection to non-negative norm physical states is to consider only states with the following quantum numbers for $l \geq 1$ (for $l = 0$ states corresponding to the continuous series are also allowed),

$$\begin{aligned} l &= (2j+1)^2 - 4p^2, \qquad & m &= 3p \\ j &= 0, 1, 2\cdots, \qquad & p &= -j, \cdots, 0, \cdots, j. \end{aligned} \tag{3.3}$$

These states correspond to the discrete series. The values taken by p suggest a hidden $SU(2)$ symmetry [7]. Our aim is to find the degeneracy for each l,j and m for these states (which may include zero norms states). Consider the quantity

$$\begin{aligned} d_r(l,n) &= \sum_{s=0}^{[s^+_{l,n}]} (-1)^s P_r\big(-\frac{1}{2}(s - s^+_{l,n})(s - s^-_{l,n})\big) \\ s^\pm_{l,n} &= -(n+\frac{1}{2}) \pm \sqrt{(n+\frac{1}{2})^2 + 2l}, \qquad n = -l, -l+1, \cdots, \end{aligned} \tag{3.4}$$

where $[a]$ denotes the integer part of the real number a. It can be easily seen, using the character formulas of the previous section and eq. (2.6), that for $r = 3$ it gives the degeneracies for the D^+ or D^- series of the $SU(1,1)_{-k}$ including the negative or zero norm states. In fact it reproduces the degeneracies [2] in fig. 2 of ref. [2]. For $r = 2$ it gives the degeneracies for the discrete series of the $SU(1,1)_{-k}/U(1)_{-k}$ parafermion theory. For $r = 1$ it gives the degeneracies of the Virasoro primary states for the discrete series in the $SU(1,1)_{-k}/U(1)_{-k}$ coset model. If we specialize (3.4) for $r = 1$ to the black hole case using (3.3) and that $n = m - j - 1$, we find the degeneracy of the black hole spectrum to be

$$d_{ljp} = 2 \sum_{s=0}^{4(j-p)+2} (-1)^s P_1(-\frac{s}{2}(s - 6j + 2p - 3)), \tag{3.5}$$

where the factor of 2 takes into account the two discrete series. Next we present a list of the possible states for the first allowed levels, along with their degeneracies.

[2] It can be proven that $d_r(l,-l) = 1$, $d_r(l,n) = 0$ for $n \leq -l-1$ and $d_r(l,n) = P_r(l)$ independent of n, for $n \geq l$.

$$
\begin{array}{llll}
\underline{l} & \underline{\frac{1}{2}d_{ljm}} & \underline{j} & \underline{m=3p} \\
1 & 1 & 0 & 0 \\
5 & 1 & 1 & -3 \\
5 & 5 & 1 & +3 \\
9 & 11 & 1 & 0 \\
9 & 1 & 2 & -6 \\
9 & 24 & 2 & +6 \\
21 & 155 & 2 & -3 \\
21 & 420 & 2 & +3 \\
25 & 704 & 2 & 0 \\
13 & 1 & 3 & -9 \\
13 & 87 & 3 & +9 \\
33 & 1179 & 3 & -6 \\
33 & 6338 & 3 & +6 \\
45 & 21148 & 3 & -3 \\
45 & 42600 & 3 & +3 \\
49 & 61545 & 3 & 0
\end{array}
\tag{3.6}
$$

Quite easily we can prove that $\frac{1}{2}d_{lj(-3j)} = 1$ and $\frac{1}{2}d_{lj(3j)} = 1 + P_1(4j+1) - P_1(2j+1)$. In both cases $l = 4j+1$. In ref. [7] another projection to non-negative norm states was proposed, in which only the principal series is considered. Then the string cannot have excitations in the black hole background. In ref. [11], this case is discussed in more detail.

To summarize, we have obtained a closed formula which gives the degeneracy of excited states of the black hole string in the discrete series. Our formula is correct as long as we include zero norm states. [3]. It would be also very interesting to generalize the $2-D$ black hole to higher anti-de-Sitter cosets $SO(d-1,2)/SO(d-1,1)$, try to find their spectrum and understand the underlying physics.

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[3] Although there are indications that negative norms are projected out [7], a complete proof of the absence of negative norm states remains to be developed.

There is another direction for the present work to develop. It has to do with the construction and classification of the modular invariants using the characters we have obtained. Certainly this by itself is a rather complicated task but, in our opinion, also worth trying since it will have implications for string phenomenology based on non-compact groups [1] and it will be the guide for a similar treatment of other non- compact groups. Work along these lines is in progress.

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