On a sphere performing linear and torsional oscillations in a viscous fluid

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Received August 27, 1986[1]

A sphere executes small-amplitude linear and torsional oscillations in a fluid at rest. The equations of motion of the fluid are solved by the method of successive approximations. Outside the boundary layer, a steady secondary flow is induced in addition to the time-varying motion.

Une sphère exécute des oscillations de faible amplitude, en ligne droite ou en torsion, dans un fluide au repos. Les équations du mouvement du fluide sont résolues par la méthode des approximations successives. À l'extérieur de la couche limite, il y a induction d'un écoulement secondaire stationnaire qui s'ajoute au mouvement variant en fonction du temps.

[Traduit par la revue]

Can. J. Phys. **66**, 576 (1988)

1. Introduction

When a body oscillates linearly at high frequencies in a fluid at rest, its oscillation generates a secondary motion as well as a time-dependent motion on the fluid. On the assumption that the boundary-layer thickness is small compared with the dimensions of the body and that the amplitude of the oscillation is small, the expression for the time-dependent part of the motion contains terms that are harmonic with respect to time, for first and higher orders. The secondary flow is steady and persists outside the boundary layer because of the action of viscosity.

Schlichting (1) was the first to initiate the study of this kind of motion. In particular, he studied the effects of transverse oscillations of a circular cylinder, both theoretically and experimentally. Using a small-amplitude expansion, he found from the first approximation a shear layer close to the body, and from the second approximation a steady secondary flow persisting even at large distances from the cylinder. Andrade (2) worked on the same topic and produced further experimental evidence. Carrier and Di Prima (3) worked on the torsional oscillations of a sphere. They expanded the velocity components in terms of the angular-displacement amplitude and solved the resulting equations up to the second approximation, so that they evaluated a correction to the torque on the sphere. Rosenblat (4) worked on the torsional oscillations of a plane. He expanded the velocity components in a power series of the amplitude and obtained first- and third-order approximations to the transverse velocity. He also found that a second-order axial flow existed outside the Stokes shear-layer, this flow being confined within a secondary layer.

The present work was undertaken on the assumption that when a sphere oscillates torsionally about an axis and simultaneously performs small-amplitude linear oscillations along the same axis, the equatorial outflow could be greatly increased. It is therefore interesting to study the flow properties that are due to this composite motion. The method of successive approximations is used to derive solutions of the equations of motion, up to second order. Each motion, torsional and transverse, produces an individual effect on the meridional components of velocity, while their effect on the azimuthal component is mutual. A jetlike flow is produced towards the sphere at its poles and away from it at the equatorial zone. The results show that the mass convected by this composite motion is much larger than that convected by purely torsional motion.

[1]Revision received February 11, 1988.

Printed in Canada / Imprimé au Canada

2. Statement of the problem

Let a sphere of radius R perform a linear oscillation along a diameter that coincides with the z axis. At the same time, it performs a torsional oscillation about the same diameter. The amplitude of both oscillations is small in comparison with the radius and the frequency is high. Let $U_\infty \cos \omega t$ be the linear velocity along the axis of oscillation, and let $\Omega_0 \cos \omega t$ be the angular velocity of the points of the sphere. Let P be a point on the surface of the sphere. P′ is another neighbouring point close to P such that PP′ is normal to the surface at P. We employ the independent variables t, ξ, ϕ, and ζ, where t denotes time, ξ is distance measured along a meridian curve from the pole Z, ϕ is the azimuthal angle, and ζ = PP′. In this orthogonal system of coordinates, ξ, ϕ, and ζ, the scale factors are $h_\xi = 1$, $h_\phi = r$, and $h_\zeta = 1$. Here, $r = R \sin(\xi/R)$ is the radius of the circle at P, which is parallel to the equatorial plane. In Fig. 1 it is r = PA. The unit vectors are $\boldsymbol{e}_\xi$, $\boldsymbol{e}_\phi$, and $\boldsymbol{e}_\zeta$. Let u, v, and w denote the (ξ, ϕ, ζ) components of velocity of the fluid. Then, the equations of the boundary layer for incompressible flow are

$$[1] \quad \frac{\partial u}{\partial t} + u\frac{\partial v}{\partial \xi} + \frac{v}{r}\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial \zeta} - \frac{v^2}{r}\frac{\mathrm{d}r}{\mathrm{d}\xi} - \nu\frac{\partial^2 u}{\partial \zeta^2} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial \xi} + \frac{V}{r}\frac{\partial U}{\partial \phi} - \frac{V^2}{r}\frac{\mathrm{d}r}{\mathrm{d}\xi}$$

$$[2] \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial \xi} + \frac{v}{r}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial \zeta} + \frac{uv}{r}\frac{\mathrm{d}r}{\mathrm{d}\xi} - \nu\frac{\partial^2 v}{\partial \zeta^2} = \frac{\partial V}{\partial t} + U\frac{\partial V}{\partial \xi} + \frac{V}{r}\frac{\partial V}{\partial \phi} + \frac{UV}{r}\frac{\mathrm{d}r}{\mathrm{d}\xi}$$

where ν is the kinematic viscosity. The equation of continuity is

$$[3] \quad \frac{\partial(ru)}{\partial \xi} + \frac{\partial v}{\partial \phi} + \frac{\partial(rw)}{\partial \zeta} = 0$$

The quantities U and V are the velocity components of the potential flow in the directions $\boldsymbol{e}_\xi$ and $\boldsymbol{e}_\phi$ at the point $(\xi, \phi, 0)$. Then $U = U_0 \cos \omega t$ and $V = 0$, where $U_0 = \frac{3}{2}U_\infty \sin(\xi/R)$.

3. Approximate solution

We now develop a solution of the previous equations employing the method of successive approximations, in which

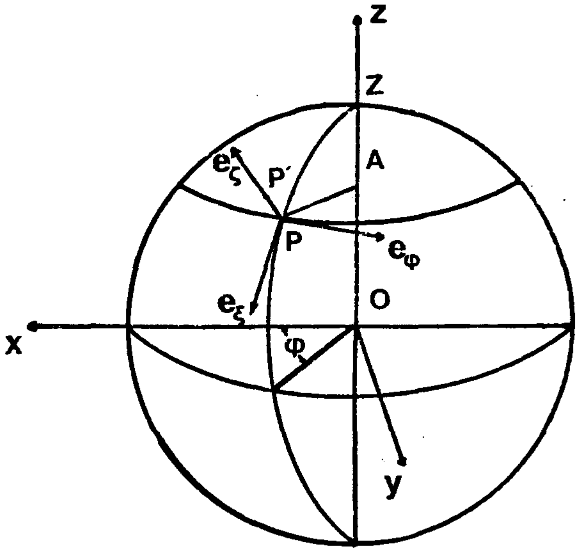

FIG. 1. Notation.

each of the unknowns is given by the following series:

$$u = \sum_{i=1}^{\infty} u_i, \qquad v = \sum_{i=1}^{\infty} v_i, \qquad w = \sum_{i=1}^{\infty} w_i$$

In the first place, the convection terms of [1] and [2] can be neglected. Therefore, the equations of motion and the equation of continuity take the form

$$[4] \quad \frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial \zeta^2} = \frac{\partial U}{\partial t}$$

$$[5] \quad \frac{\partial v_1}{\partial t} - \nu \frac{\partial^2 v_1}{\partial \zeta^2} = 0$$

$$[6] \quad \frac{\partial (r u_1)}{\partial \xi} + \frac{\partial v_1}{\partial \phi} + \frac{\partial (r w_1)}{\partial \zeta} = 0$$

with boundary conditions

$$u_1 = U_0 \cos \omega t, \qquad \zeta = \infty$$
$$u_1 = 0, \qquad \zeta = 0$$
$$v_1 = 0, \qquad \zeta = \infty$$
$$v_1 = \Omega_0 r \cos \omega t, \qquad \zeta = 0$$

We now employ the complex notation

$$U(\zeta, t) = U_0(\zeta)\, \mathrm{e}^{i\omega t}, \qquad \Omega = \Omega_0\, \mathrm{e}^{i\omega t}$$

and we introduce the nondimensional coordinate $\eta = \zeta\sqrt{\omega/\nu}$. Seeking now solutions of the form

$$u_1 = U f_1(\eta), \qquad v_1 = \Omega R g_1(\eta) \sin(\xi/R)$$

we derive the following equations:

$$f_1'' - i f_1 = -i$$
$$g_1'' - i g_1 = 0$$

with boundary conditions

$$f_1(0) = 0, \qquad f_1(\infty) = 1, \qquad g_1(0) = 1, \qquad g_1(\infty) = 0$$

The solutions are

$$[7] \quad u_1 = U_0 \cos \omega t - U_0\, \mathrm{e}^{-p} \cos(\omega t - p)$$

$$[8] \quad v_1 = \Omega_0 R\, \mathrm{e}^{-p} \cos(\omega t - p) \sin(\xi/R)$$

where $p = \eta/\sqrt{2}$. Hence, from [6] it follows that

$$[9] \quad w_1 = -3\sqrt{\frac{\nu}{\omega}}\, \frac{U_\infty}{R} \cos\frac{\xi}{R} \left\{ \eta \cos \omega t + \frac{\sqrt{2}}{2} \times \mathrm{e}^{-p}[\sin(\omega t - p) + \cos(\omega t - p)] - \frac{\sqrt{2}}{2}(\sin \omega t + \cos \omega t) \right\}$$

For large values of η, w_1 tends to

$$[10] \quad w_1 = -3\sqrt{\frac{\nu}{\omega}}\, \frac{U_\infty}{R} \cos\frac{\xi}{R} \left[\eta \cos \omega t - \cos\left(\omega t - \frac{\pi}{4}\right) \right]$$

To take into account higher order terms, we put $u = u_1 + u_2$, $v = v_1 + v_2$, and $w = w_1 + w_2$ in [1]–[3]. Hence

$$[11] \quad \frac{\partial u_2}{\partial t} - \nu \frac{\partial^2 u_2}{\partial \zeta^2} = U \frac{\partial U}{\partial \xi} - u_1 \frac{\partial u_1}{\partial \xi} - w_1 \frac{\partial u_1}{\partial \zeta} + \frac{v_1^2}{r}\frac{\mathrm{d}r}{\mathrm{d}\xi}$$

$$[12] \quad \frac{\partial v_2}{\partial t} - \nu \frac{\partial^2 v_2}{\partial \zeta^2} = -u_1 \frac{\partial v_1}{\partial \xi} - w_1 \frac{\partial v_1}{\partial \zeta} - \frac{u_1 v_1}{r}\frac{\mathrm{d}r}{\mathrm{d}\xi}$$

$$[13] \quad \frac{\partial (r u_2)}{\partial \xi} + \frac{\partial v_2}{\partial \phi} + \frac{\partial (r w_2)}{\partial \zeta} = 0$$

The resulting form of the right-hand side members of [11] and [12], after the substitution of U, u_1, v_1, and w_1 by their values, implies that it is convenient to seek solutions having the form

$$[14] \quad u_2 = a_1 f_{21}(p) + a_2 f_{22}(p) + [a_3 f_{23}(p) + a_4 f_{24}(p)] \times \mathrm{e}^{2i\omega t}$$

$$[15] \quad v_2 = a_5 g_{21}(p) + a_6 g_{22}(p)\, \mathrm{e}^{2i\omega\tau}$$

where

$$a_1 = a_3 = \frac{1}{2\omega} U_0 \frac{\partial U_0}{\partial \xi}$$

$$a_2 = a_4 = \frac{1}{2\omega} \Omega_0^2 R \cos\frac{\xi}{R} \sin\frac{\xi}{R}$$

$$a_5 = a_6 = \frac{3}{\omega} U_\infty \Omega_0 \sin\frac{\xi}{R} \cos\frac{\xi}{R}$$

It follows then that

$$[16] \quad f_{21}'' = r_1$$

$$[17] \quad f_{22}'' = r_2$$

$$[18] \quad f_{23}'' - 4i f_{23} = r_3$$

$$[19] \quad f_{24}'' - 4i f_{24} = r_4$$

$$[20] \quad g_{21}'' = r_5$$

$$[21] \quad g_{22}'' - 4i g_{22} = r_6$$

where

$$r_1 = 2\,\mathrm{e}^{-p}(2\sin p - 2\cos p + \mathrm{e}^{-p} - 2p\cos p - 2p\sin p)$$

$$r_2 = -2\,\mathrm{e}^{-2p}$$

$$r_3 = -4p\,\mathrm{e}^{-p}(\cos p - i\sin p + i\cos p + \sin p) - 2\,\mathrm{e}^{-2p}\cos 2p$$

$$r_4 = -2(\cos 2p - i \sin 2p)$$
$$r_5 = e^{-p}[\cos p - \sin p + p(\cos p + \sin p) - e^{-p}]$$
$$r_6 = p\, e^{-p}[\cos p(1+i) + \sin p(1-i)]$$

The boundary conditions are

$$f_{2i}(0) = g_{2j}(0) = 0, \qquad f_{2i}(\infty) = g_{2j}(\infty) = \text{finite}, \qquad i = 1, 2, 3, 4, \qquad j = 1, 2$$

Solving [13] for w_2, one derives

$$[22] \quad w_2 = -\sqrt{\frac{2\nu}{\omega}}\,[b_{21}h_{21} + b_{22}h_{22} + (b_{23}h_{23} + b_{24}h_{24}) \times e^{2i\omega t}]$$

where

$$b_{2i} = \frac{a_i}{r}\cos\frac{\xi}{R} + \frac{\partial a_i}{\partial \xi} \qquad h_{2i} = \int_0^p f_{2i}\,dp, \qquad i = 1, 2, 3, 4$$

The expressions for f_{2i}, g_{2j}, and h_{2i} are given in the Appendix.

4. Results and discussion

One can therefore deduce that at large distances from the sphere, a steady flow is induced. This flow is generated by viscous forces that convect the effect of the periodic motion away from the oscillating sphere. The velocity components of this permanent flow are

$$[23] \quad u_{2\infty} = \frac{1}{8R\omega}\left(\frac{45}{4}U_\infty^2 + \Omega_0^2R^2\right)\sin 2\theta$$

$$[24] \quad v_{2\infty} = \frac{3}{2R\omega}U_\infty\Omega_0 R \sin 2\theta$$

$$[25] \quad w_{2\infty} = -\frac{1}{8\omega R^2}\sqrt{\frac{2\nu}{\omega}}\left[\frac{9}{4}U_\infty^2(21 + 5\sqrt{2}\eta_\infty) + \Omega_0^2R^2(\eta_\infty\sqrt{2} - 1)\right](2\cos^2\theta - \sin^2\theta)$$

where $\theta = \xi/R$ and η_∞ is the boundary-layer width. On the upper half of the sphere, the velocity components $u_{2\infty}$ and $v_{2\infty}$ are positive, while on the lower half they are negative. This shows that on the upper half, the flow generated by these two components is balanced by a reverse flow in the lower half. As regards the radial component $w_{2\infty}$, the effects are more complicated. There are three spherical zones: one equatorial and two surrounding the poles on either side of the former, with opposite flow properties. The equatorial zone extends between $\theta = 55°$ and 125°. There is an outflow at the equatorial zone and an inflow at the polar zones. The velocity-vector diagram and the direction of the velocity components relative to an observer at rest are sketched in Figs. 2 and 3. Thus, on the upper half there is a west-to-east drift, inward in the polar zones and outward in the equatorial zone. In the lower half there is east-to-west drift. Finally, on the poles the flow is axial inwards; on the equator it is radial outwards; and at $\theta = 55°$ and 125° there is no radial component: the velocity there is tangential.

With respect to the pumping effects that are suggested by [25], it can be seen that the linear oscillation increases the pumping ability in relation to the purely torsional motion by a factor in the order of $45x_0^2/R^2\varepsilon^2$, where x_0 and ε are the amplitudes of the linear and torsional oscillation respectively.

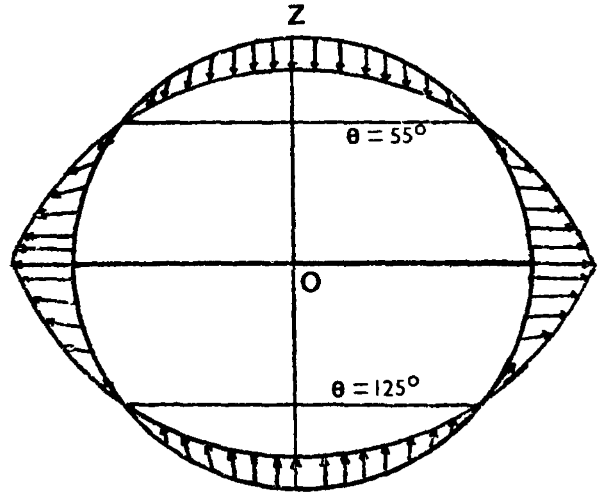

FIG. 2. Velocity vector diagram on the meridional planes $\phi = 0$ and π.

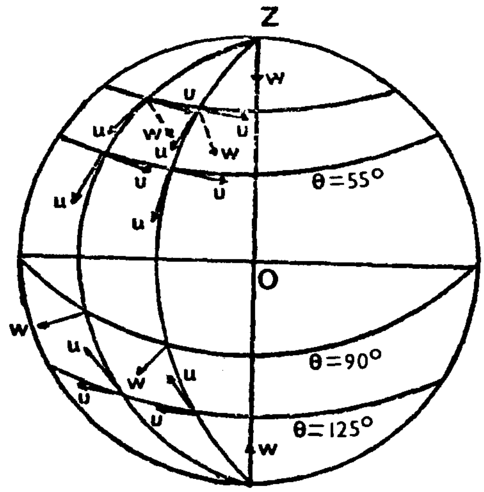

FIG. 3. Direction of velocity components.

5. Conclusions

When a sphere undergoes small-amplitude reciprocating and torsional oscillations in a viscous fluid, then the Reynolds number of the flow $U_\infty^2/\omega\nu$ can be assumed to be small; thus the Navier–Stokes equations can be approximated by appropriate linear forms. It has been proven that outside the boundary layer, a stationary flow is induced in addition to the time-dependent flow. This permanent effect is associated with the action of viscosity. The tangential velocity components generate opposite and balanced flows at the upper and lower hemispheres. The radial component generates both inflow and outflow at three distinct spherical zones.

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Appendix

Functions pertaining to the velocity components u_2, v_2, and w_2:

$$f_{21} = 6\,e^{-p}\sin p + \frac{1}{2}\,e^{-2p} + 2p\,e^{-p}(\sin p - \cos p) + 2\,e^{-p}\cos p - \frac{5}{2}$$

$$f_{22} = \frac{1}{2}\,(1 - e^{-2p})$$

$$f_{23} = 4\,e^{-(i+1)p} - 2p(1+i)\,e^{-(i+1)p} + \frac{1}{6}\,e^{-2p}\sin 2p - 4\,e^{-\sqrt{2}(i+1)p}$$

$$f_{24} = \frac{1}{4}\,(1-i)\,e^{-2ip} - \frac{1}{4}\,(1-i)\,e^{-\sqrt{2}(1+i)p}$$

$$g_{21} = \frac{1}{2}\,p\,e^{-p}(\cos p - \sin p) - \frac{1}{2}\,e^{-p}(\cos p + 3\sin p) - \frac{1}{2}\,e^{-2p} + 1$$

$$g_{22} = -i\,e^{-(i+1)p} - \frac{1}{2}\,p\,e^{-(i+1)p} + \frac{1}{2}\,ip\,e^{-(i+1)p} + i\,e^{-\sqrt{2}(i+1)p}$$

$$h_{21} = -5\,e^{-p}\cos p - 3\,e^{-p}\sin p - 2p\,e^{-p}\sin p - \frac{1}{4}\,e^{-2p} + \frac{5}{2}\,p + \frac{21}{4}$$

$$h_{22} = \frac{1}{4}\,e^{-2p} + \frac{1}{2}\,p - \frac{1}{4}$$

$$h_{23} = (1 - i + 2p)\,e^{-(i+1)p} - \frac{1}{24}\,e^{-2p}(\cos 2p + \sin 2p) + (1-i)\sqrt{2}\,e^{-\sqrt{2}p(1+i)} + (i-1)(1+\sqrt{2}) + \frac{1}{24}$$

$$h_{24} = \frac{1}{8}\,(i+1)\,e^{-2ip} - \frac{\sqrt{2}}{8}\,i\,e^{-\sqrt{2}p(i+1)} - \frac{1}{8}\,(i+1) + \frac{\sqrt{2}}{8}\,i$$