

# SPECIAL FUNCTIONS and POLYNOMIALS

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Many of the special functions and polynomials are constructed along standard procedures In this short survey we list the most essential ones.

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# 1 Legendre Polynomials $P_\ell(x)$ .

Differential Equation:

$$(1 - x^2) P_\ell''(x) - 2x P_\ell'(x) + \ell(\ell + 1) P_\ell(x) = 0 ,$$

or

$$\frac{d}{dx}(1 - x^2) \frac{d}{dx} P_\ell(x) + \ell(\ell + 1) P_\ell(x) = 0 . \quad (1.1)$$

Generating function:

$$\sum_{\ell=0}^{\infty} P_\ell(x) t^\ell = (1 - 2xt + t^2)^{-\frac{1}{2}} \quad \text{for } |t| < 1, |x| \leq 1. \quad (1.2)$$

Orthonormality:

$$\int_{-1}^1 P_\ell(x) P_{\ell'}(x) dx = \frac{2}{2\ell + 1} \delta_{\ell\ell'} , \quad (1.3)$$

$$\sum_{\ell=0}^{\infty} P_\ell(x) P_\ell(x') (2\ell + 1) = 2\delta(x - x') . \quad (1.4)$$

Expressions for  $P_\ell(x)$  :

$$P_\ell(x) = \frac{1}{2^\ell} \sum_{\nu=0}^{[\ell/2]} \frac{(-1)^\nu (2\ell - 2\nu)!}{\nu! (\ell - \nu)! (\ell - 2\nu)!} x^{\ell-2\nu} \quad (1.5)$$

$$= \frac{1}{\ell! 2^\ell} \left( \frac{d}{dx} \right)^\ell (x^2 - 1)^\ell , \quad (1.6)$$

$$= \frac{1}{\pi} \int_0^\pi (x + \sqrt{x^2 - 1} \cos \varphi)^\ell d\varphi . \quad (1.7)$$

Recurrence relations:

$$\ell P_{\ell-1} - (2\ell + 1)x P_\ell + (\ell + 1) P_{\ell+1} = 0 ;$$

$$P_\ell = x P_{\ell-1} + \frac{x^2 - 1}{\ell} P'_{\ell-1} ;$$

$$x P'_\ell - \ell P_\ell = P'_{\ell-1} ;$$

$$x P'_\ell + (\ell + 1) P_\ell = P'_{\ell+1} ;$$

$$\frac{d}{dx} [P_{\ell+1} - P_{\ell-1}] = (2\ell + 1) P_\ell . \quad (1.8)$$

Examples:

$$P_0 = 1 , \quad P_1 = x , \quad P_2 = \frac{1}{2}(3x^2 - 1) , \quad P_3 = \frac{1}{2}x(5x^2 - 3) . \quad (1.9)$$

## 2 Associated Legendre Functions $P_\ell^m(x)$ .

Differential equation:

$$(1 - x^2) P_\ell^m(x)'' - 2x P_\ell^m(x)' + \left( \ell(\ell + 1) - \frac{m^2}{1 - x^2} \right) P_\ell^m(x) = 0 . \quad (2.1)$$

Generating function:

$$\sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \frac{P_\ell^m(x) z^m y^\ell}{m!} = \left[ 1 - 2y \left( x + z\sqrt{1 - x^2} \right) + y^2 \right]^{-\frac{1}{2}} . \quad (2.2)$$

Orthogonality:

$$\int_{-1}^1 P_\ell^m(x) P_{\ell'}^m(x) dx = \frac{2}{2\ell + 1} \frac{(\ell + m)!}{(\ell - m)!} \delta_{\ell\ell'} , \quad (\ell, \ell' \geq m) . \quad (2.3)$$

$$\sum_{\ell=m}^{\infty} (2\ell + 1) \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(x) P_\ell^m(x') = 2\delta(x - x') , \quad (|x| < 1 \text{ and } |x'| < 1) . \quad (2.4)$$

Expressions for  $P_\ell^m(x)$ <sup>1</sup>:

$$P_\ell^m(x) = (1 - x^2)^{\frac{1}{2}m} \left( \frac{d}{dx} \right)^m P_\ell(x) . \quad (2.5)$$

$$P_\ell^m(x) = \frac{(\ell + m)!}{\ell! \pi} (-1)^{m/2} \int_0^\pi \left( x + \sqrt{x^2 - 1} \cos \varphi \right)^\ell \cos m\varphi d\varphi . \quad (2.6)$$

Recurrence relations:

$$P_\ell^{m+1} - \frac{2mx}{\sqrt{1 - x^2}} P_\ell^m + \{ \ell(\ell + 1) - m(m - 1) \} P_\ell^{m-1} = 0 \quad (2.7)$$

$$\sqrt{1 - x^2} P_\ell^{m+1}(x) = (1 - x^2) P_\ell^m(x)' + mx P_\ell^m(x) ,$$

$$(2\ell + 1)x P_\ell^m = (\ell + m) P_{\ell-1}^m + (\ell + 1 - m) P_{\ell+1}^m , \quad (2.8)$$

$$x P_\ell^m = P_{\ell-1}^m - (\ell + 1 - m) \sqrt{1 - x^2} P_\ell^{m-1} ,$$

$$P_{\ell+1}^m - P_{\ell-1}^m = (2\ell + 1) P_\ell^{m-1} \sqrt{1 - x^2} , \quad (2.9)$$

and various others.

Examples:

$$\begin{aligned} P_1^1 &= \sqrt{1 - x^2} , & P_2^2 &= 3(1 - x^2) , \\ P_2^1 &= 3x\sqrt{1 - x^2} , & P_3^2 &= 15x(1 - x^2) . \end{aligned} \quad (2.10)$$

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<sup>1</sup>Note that some authors define  $P_\ell^m(x)$  with a factor  $(-1)^m$ , giving  $P_\ell^m(x) = (-1)^m (1 - x^2)^{\frac{1}{2}m} \left( \frac{d}{dx} \right)^m P_\ell(x)$ . Obviously this minus sign propagates to the generating function, the recurrence relations and the explicit examples, when  $m$  is odd.

### 3 Bessel $J_n(x)$ and Hankel $H_n(x)$ functions.

Differential equation (for both  $J_n$  and  $H_n$ ):

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0. \quad (3.1)$$

Generating function (if  $n$  integer):

$$\sum_{n=-\infty}^{\infty} J_n(\alpha x) \left(\frac{s}{\alpha}\right)^n = e^{\frac{x}{2}(s - \frac{\alpha^2}{s})}, \quad (3.2)$$

$$J_{-n} = (-1)^n J_n.$$

Orthogonality:

$$\int_0^{\infty} \xi J_n(\alpha \xi) J_n(\beta \xi) d\xi = \frac{1}{\alpha} \delta(|\alpha| - |\beta|). \quad (3.3)$$

$$\int_0^a \xi J_n(\alpha \xi) J_n(\beta \xi) d\xi = \frac{a^2}{2} \{J_{n+1}(\alpha a)\}^2 \delta_{\alpha\beta}. \quad (3.4)$$

if in the 2<sup>nd</sup> relation  $\alpha, \beta$  are roots of the equation  $J_n(\alpha \xi) = 0$ .

Expressions for  $J_n(x)$  (for  $n$  integer):

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{n+2k} = \frac{1}{2\pi i} \left(\frac{x}{2}\right)^n \oint t^{-n-1} dt e^{t-x^2/4t}. \quad (3.5)$$

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin \theta) d\theta. \quad (3.6)$$

Recurrence relations (for both  $J_n$  and  $H_n$ ):

$$\frac{d}{dx} \{x^n J_n(x)\} = x^n J_{n-1}(x);$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x);$$

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) \quad (3.7)$$

Relations between Hankel and Bessel functions:

$$H_n^{(1)}(x) = \frac{i}{\sin n\pi} (e^{-n\pi i} J_n(x) - J_{-n}(x));$$

$$H_n^{(2)}(x) = \frac{-i}{\sin n\pi} (e^{n\pi i} J_n(x) - J_{-n}(x)), \quad (3.8)$$

so that

$$J_n(x) = \frac{1}{2}(H_n^{(1)}(x) + H_n^{(2)}(x));$$

$$J_{-n}(x) = \frac{1}{2}(e^{n\pi i} H_n^{(1)}(x) + e^{-n\pi i} H_n^{(2)}(x)). \quad (3.9)$$

## 4 Spherical Bessel Functions $j_\ell(x)$ .

Differential equation:

$$(xj_\ell)'' + \left(x - \frac{\ell(\ell+1)}{x}\right)j_\ell = 0. \quad (4.1)$$

Generating Function:

$$\sum_{\ell=0}^{\infty} \frac{j_\ell(x) t^\ell}{\ell!} = j_0(\sqrt{x^2 - 2xt}). \quad (4.2)$$

Orthogonality:

$$\int_0^\infty x^2 j_\ell(\alpha x) j_\ell(\beta x) dx = \frac{\pi}{2\alpha} \delta(\alpha - \beta). \quad (4.3)$$

$$\int_{-\infty}^{\infty} j_\ell(x) j_{\ell'}(x) dx = \frac{\pi}{2\ell+1} \delta_{\ell\ell'}. \quad (4.4)$$

Expressions for  $j_\ell$ :

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+\frac{1}{2}}(x) = (-1)^\ell x^\ell \left(\frac{d}{xdx}\right)^\ell \frac{\sin x}{x}, \quad (4.5)$$

$$\begin{aligned} j_\ell(x) &= \frac{x^\ell}{2^{\ell+1} \ell!} \int_{-1}^1 e^{ixs} (1-s^2)^\ell ds \\ &= \frac{2^\ell \ell!}{(2\ell+1)!} x^\ell \left(1 - \frac{1}{1!(\ell+\frac{3}{2})} \left(\frac{x}{2}\right)^2 + \frac{1}{2!(\ell+\frac{3}{2})(\ell+\frac{5}{2})} \left(\frac{x}{2}\right)^4 - \dots\right). \end{aligned} \quad (4.6)$$

Recurrence relations:

$$j_{\ell+1} = \frac{\ell}{x} j_\ell - j'_\ell = \frac{2\ell+1}{x} j_\ell - j_{\ell-1}. \quad (4.7)$$

Examples:

$$\begin{aligned} j_0(x) &= \frac{\sin x}{x}; \\ j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x}; \\ j_2(x) &= \frac{3 \sin x}{x^3} - \frac{3 \cos x}{x^2} - \frac{\sin x}{x}. \end{aligned} \quad (4.8)$$

## 5 Hermite Polynomials $H_n(x)$ .

Differential equation:

$$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0 ,$$

or

$$\frac{d^2}{dx^2} \left( H_n(x) e^{-\frac{1}{2}x^2} \right) + (2n - x^2 + 1) H_n(x) e^{-\frac{1}{2}x^2} = 0 . \quad (5.1)$$

Generating function:

$$\sum_{n=0}^{\infty} H_n(x) s^n / n! = e^{-s^2 + 2sx} . \quad (5.2)$$

Orthogonality:

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = 2^n n! \sqrt{\pi} \delta_{nm} \quad (5.3)$$

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) / (2^n n!) = \sqrt{\pi} \delta(x - y) e^{x^2} . \quad (5.4)$$

More general:

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) s^n / (2^n n!) = \frac{1}{\sqrt{1-s^2}} \exp \left( \frac{-s^2(x^2 + y^2) + 2sxy}{1-s^2} \right) . \quad (5.5)$$

Expressions for  $H_n$  :

$$H_n(-x) = (-1)^n H_n(x); \quad (5.6)$$

$$H_n(x) = (-1)^n e^{x^2} \left( \frac{d}{dx} \right)^n e^{-x^2} ; \quad (5.7)$$

$$H_n(x) = (-1)^{n/2} n! \sum_{k=0}^{n/2} (-1)^k \frac{(2x)^{2k}}{(2k)! (\frac{1}{2}n - k)!} , \quad \text{if } n \text{ even,}$$

$$H_n(x) = (-1)^{\frac{n-1}{2}} n! \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \frac{(2x)^{2k+1}}{(2k+1)! \left( \frac{n-1}{2} - k \right)!} , \quad \text{if } n \text{ odd.} \quad (5.8)$$

Recurrence relations:

$$\frac{d^m H_n(x)}{dx^m} = \frac{2^m n!}{(n-m)!} H_{n-1}(x) , \quad (5.9)$$

$$x H_n(x) = \frac{1}{2} H_{n+1}(x) + n H_{n-1}(x) , \quad (5.10)$$

$$H_n(x) = \left( 2x - \frac{d}{dx} \right) H_{n-1}(x) . \quad (5.11)$$

Examples:

$$H_0(x) = 1 , \quad H_1(x) = 2x , \quad H_2(x) = 4x^2 - 2 . \quad (5.12)$$

## 6 Laguerre Polynomials $L_n(x)$ .

Differential equation:

$$x L_n''(x) + (1 - x) L_n'(x) + n L_n(x) = 0 . \quad (6.1)$$

Generating function:<sup>2</sup>

$$\sum_{n=0}^{\infty} L_n(x) z^n = \frac{1}{1 - z} e^{\frac{-xz}{1-z}} . \quad (6.2)$$

Orthogonality:

$$\int_0^{\infty} L_n(x) L_m(x) e^{-x} dx = \delta_{nm} . \quad (6.3)$$

Expressions for  $L_n$  :

$$\begin{aligned} L_n(x) &= \frac{e^x}{n!} \left( \frac{d}{dx} \right)^n (x^n e^{-x}) \\ &= \frac{(-1)^n}{n!} \left( x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2(n-1)^2}{2!} x^{n-2} - \dots + (-1)^n n! \right) . \end{aligned} \quad (6.4)$$

Recurrence relation:

$$\begin{aligned} (1 + 2n - x) L_n - n L_{n-1} - (n + 1) L_{n+1} &= 0 ; \\ x L_n'(x) &= n L_n(x) - n L_{n-1}(x) . \end{aligned} \quad (6.5)$$

Examples:

$$\begin{aligned} L_0(x) &= 1 ; \\ L_1(x) &= 1 - x ; \\ L_2(x) &= \frac{1}{2!} (x^2 - 4x + 2) . \end{aligned} \quad (6.6)$$

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<sup>2</sup>It's important to note that sometimes different definitions are used for the Laguerre and Associated Laguerre polynomials, where the Generating Function has the form:  $\sum_{n=0}^{\infty} L_n(x) z^n / n! = \frac{1}{1-z} e^{\frac{-xz}{1-z}}$ . In this case the Expressions given for  $L_n$  should be multiplied by  $n!$ .

## 7 Associated Laguerre Polynomials $L_n^k(x)$ .

Differential equation:

$$x L_n^{k''} + (k + 1 - x) L_n^{k'} + n L_n^k = 0 . \quad (7.1)$$

Generating function:

$$\sum_{n=0}^{\infty} L_n^k(x) z^n = \frac{1}{(1-z)^{k+1}} e^{\frac{-xz}{1-z}} . \quad (7.2)$$

$$\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{L_n^k(x) z^n u^k}{k!} = \frac{1}{1-z} \exp\left(\frac{-xz+u}{1-z}\right) . \quad (7.3)$$

Orthogonality:

$$\int_0^{\infty} L_n^k(x) L_m^k(x) x^k e^{-x} dx = \frac{(n+k)!}{n!} \delta_{nm} . \quad (7.4)$$

Expressions for  $L_n^k$  :

$$L_n^k(x) = (-1)^k \left(\frac{d}{dx}\right)^k L_{n+k}(x) . \quad (7.5)$$

$$L_n^k(x) = \frac{e^x x^{-k}}{n!} \left(\frac{d}{dx}\right)^n (x^{n+k} e^{-x}) . \quad (7.6)$$

Recurrence relation:

$$\begin{aligned} L_{n-1}^k(x) + L_n^{k-1}(x) &= L_n^k(x) ; \\ x L_n^{k'}(x) &= n L_n^k(x) - (n+k) L_{n-1}^k(x) . \end{aligned} \quad (7.7)$$

Examples:

$$\begin{aligned} L_0^k(x) &= 1 ; \\ L_1^k(x) &= -x + k + 1 ; \\ L_2^k(x) &= \frac{1}{2} \left[ x^2 - 2(k+2)x + (k+1)(k+2) \right] ; \\ L_3^k(x) &= \frac{1}{6} \left[ -x^3 + 3(k+3)x^2 - 3(k+2)(k+3)x + (k+1)(k+2)(k+3) \right] . \end{aligned} \quad (7.8)$$



## 8 Tschebyscheff<sup>3</sup> Polynomials $T_n(x)$ .

Differential equation:

$$(1-x^2)\frac{d^2}{dx^2}T_n(x) - x\frac{d}{dx}T_n(x) + n^2T_n(x) = 0. \quad (8.1)$$

Generating function:

$$\sum_{n=0}^{\infty} T_n(x) y^n = \frac{1-xy}{1-2xy+y^2}. \quad (8.2)$$

Symmetry relation:

$$T_n(x) = T_{-n}(x). \quad (8.3)$$

Orthogonality:

$$\int_{-1}^1 \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} \frac{1}{2}\pi\delta_{nm} & m, n \neq 0 \\ \pi & m = n = 0 \end{cases} \quad (8.4)$$

Expression for  $T_n$ :

$$T_n(x) = \cos(n \cos^{-1} x) \quad (8.5)$$

$$T_n(x) = \frac{1}{2} \left[ \left\{ x + i\sqrt{1-x^2} \right\}^n + \left\{ x - i\sqrt{1-x^2} \right\}^n \right]. \quad (8.6)$$

Recurrence relation:

$$T_{n+1} - 2xT_n(x) + T_{n-1} = 0 \quad (8.7)$$

$$(1-x^2)T_n'(x) = -nxT_n(x) + nT_{n-1}(x). \quad (8.8)$$

Examples:

$$\begin{aligned} T_0(x) &= 1; \\ T_1(x) &= x; \\ T_2(x) &= 2x^2 - 1; \\ T_3(x) &= 4x^3 - 3x. \end{aligned} \quad (8.9)$$

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<sup>3</sup>Transliterations Chebyshev and Tchebicheff also occur.

## 9 Remark.

All of the functions discussed here are special cases of “hypergeometric functions”  ${}_mF_n(a_1, a_2, \dots, a_m; b_1, b_2, \dots, b_n; z)$  defined by:

$${}_mF_n(a_1, a_2, \dots, a_m; b_1, b_2, \dots, b_n; z) = \sum_{r=p}^{\infty} \frac{(a_1)_r (a_2)_r \dots (a_m)_r z^r}{(b_1)_r (b_2)_r \dots (b_n)_r r!}, \quad (9.1)$$

where

$$(a)_r \equiv \frac{\Gamma(a+r)}{\Gamma(a)}; \quad r \text{ a positive integer.} \quad (9.2)$$

Differential equations:

$m = n = 1$ :

$$z \left( \frac{d}{dz} \right)^2 {}_1F_1 + (b-z) \frac{d}{dz} {}_1F_1 - a {}_1F_1 = 0. \quad (9.3)$$

$m = 2, n = 1$ :

$$z(1-z) \left( \frac{d}{dz} \right)^2 {}_2F_1 + (c - (a+b+1)z) \frac{d}{dz} {}_2F_1 - ab {}_2F_1 = 0. \quad (9.4)$$

We have:

$$P_\ell(x) = {}_2F_1 \left( -\ell, \ell+1; 1; \frac{1-x}{2} \right); \quad (9.5)$$

$$P_\ell^m(x) = \frac{(\ell+m)! (1-x^2)^{m/2}}{(\ell-m)! 2^m m!} {}_2F_1 \left( m-\ell, m+\ell+1; m+1; \frac{1-x}{2} \right); \quad (9.6)$$

$$J_n(x) = \frac{e^{-ix}}{n!} \left( \frac{x}{2} \right)^n {}_1F_1 \left( n + \frac{1}{2}; 2n+1; 2ix \right); \quad (9.7)$$

$$H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} {}_1F_1 \left( -n; \frac{1}{2}; x^2 \right); \quad (9.8)$$

$$H_{2n+1}(x) = (-1)^n \frac{2(2n+1)!}{n!} x {}_1F_1 \left( -n; \frac{3}{2}; x^2 \right); \quad (9.9)$$

$$L_n(x) = {}_1F_1(-n; 1; x); \quad (9.10)$$

$$L_n^k(x) = \frac{\Gamma(n+k+1)}{n! \Gamma(k+1)} {}_1F_1(-n; k+1; x); \quad (9.11)$$

$$T_n(x) = {}_2F_1 \left( -n, n; \frac{1}{2}; \frac{1-x}{2} \right). \quad (9.12)$$

## References

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